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INTEGRABLE DYNAMICAL SYSTEMS AND KAC-MOODY ALGEBRAS

A.D.W.B. Crumey
Blackett Laboratory
Imperial College
Prince Consort Road,
London SW7 2BZ

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ABSTRACT

It is shown how to construct realizations of centre-free Kac-Moody algebras as hierarchies of 1+1 dimensional classical dynamical systems. The equations of motion (which are, in general, non-local) have Hamiltonians which form realizations of the same algebra. A Cartan subalgebra provides infinitely many conserved quantities in involution, while a sub-class of the step operators (which may be interpreted as generators of translations in "internal dimensions") enable the systems to be linearized. The systems can be regarded as having a "gauge symmetry" which includes momentum. Examples include the sine-Gordon, Burgers and non-linear Schrodinger equations.

1. INTRODUCTION

Ever since the work of Zakharov and Shabat [1], it has been clear that a wide range of physically important non-linear evolution equations in 1+1 dimensions may be written as the compatibility condition of a pair of linear equations, i.e. as a zero curvature equation

$$[\partial_x + A_x, \partial_t + A_t] = 0 \tag{1.1}$$

where the "gauge potentials" A_x, A_t are matrices, usually written as polynomials (or rational functions) in a complex parameter λ (which does not appear in the equation of motion) with coefficients in a Lie algebra $\underset{\sim}{g}$.

For a given equation of motion, it may be very difficult (and will usually be impossible) to obtain a zero-curvature representation. A more reasonable approach is to try and classify the sort of equations which can arise from (1.1). If A_x is fixed, then one may seek A_t to arbitrary order by equating powers of λ to zero in (1.1). This will give rise to a hierarchy of equations of motion. The problem of finding A_t is made easier by the invariance of (1.1) under a "gauge transformation"

$$A_\mu \to \gamma^{-1} A_\mu \gamma + \gamma^{-1}\gamma_\mu \equiv a_\mu \tag{1.2}$$

(where $\gamma_\mu \equiv \partial_\mu \gamma$). The matrix coefficients of γ belong to the Lie group G of $\underset{\sim}{g}$. It may be possible to construct a transformation $A_x \to a_x$ which allows the zero curvature condition (1.1) to be solved for a_t. Then A_t is obtained via the inverse transformation.

In [2,3], this was carried out for the generalized non-linear Schrodinger hierarchy [4], in which the gauge potentials are polynomials with coefficients in a symmetric Lie algebra $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$. By transforming A_x to a constant element, it was possible to obtain a closed expression for A_t to any order in powers of λ, and so the commutation properties of the hierarchy of evolution operators (∂_t) could be investigated. They were found to provide a realization of the centre free Kac-Moody algebra $\underset{\sim}{k} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. It was then possible to construct a further hierarchy of equations so as to obtain the full algebra $g \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. The evolution operators corresponding to the subspace $\underset{\sim}{m}$ were found not to commute with the translation operator ∂_x, and were regarded as translations in "internal" dimensions, in analogy with the situation in gauge field theories [5]. The Hamiltonians of the evolution operators were found to provide a classical "current algebra" realization of the same algebra. (The central term which arose will, in this paper, be seen to vanish). The Hamiltonians corresponding to the subalgebra $\underset{\sim}{k}$ could then be regarded as conserved quantities for the generalized non-linear Schrodinger equation, while those corresponding to $\underset{\sim}{m}$ could be used to linearize the equation of motion. The system was thus shown to be completely integrable.

In the present work, constant gauge potentials will be used to obtain other polynomial pairs (A_x, A_t) associated with integrable dynamical systems. One begins with a non-commutative version of the zero curvature condition

$$\partial_{n_1,g_1} A_{n_2,g_2} - \partial_{n_2,g_2} A_{n_1,g_1} + [A_{n_1,g_1}, A_{n_2,g_2}] = A_{n_1+n_2,[g_1,g_2]} \quad (1.3)$$

where $\partial_{n,g}$ $(n \in \mathbb{Z},\ g \in g)$ form a centre free Kac-Moody algebra, and are regarded as operators of differentiation with respect to parameters $t_{n,g}$. One of these parameters (with g an element of a Cartan subalgebra $\underset{\sim}{h}$ of g) will be chosen as "space", labelled x. Those operators $\partial_{n,k}$ which commute with ∂_x may be regarded as defining a hierarchy of evolution equations, while those which do not are regarded as "internal" translations. The Hamiltonians will be shown to provide a realization of the same Kac-Moody algebra, and the dynamical systems defined by $\partial_{n,h}$ $(h \in \underset{\sim}{h})$ will be completely integrable. The equations of motion are, in general, non-local. The construction of such systems will be described in Section 2. The key ingredient is the gauge transformation which takes constant solutions of (1.3) to dynamical ones. This is obtained from a "gauge equation", which is discussed in detail in Section 3. In particular, it is shown that for any choice of x, it is always possible to construct a hierarchy of integrable systems. Also, if A_x is expressed locally in terms of dynamical fields and their derivatives at a point, then there will always be a hierarchy of local equations of motion (and hence there will be local conserved quantities). In Sections 4 and 5, some examples will be given of equations of motion which arise from the construction.

2. METHOD

Let g be a semisimple Lie algebra. The "loop algebra" g' is the centre-free Kac-Moody algebra $g \otimes \mathbb{C}[\lambda,\lambda^{-1}]$, where $\mathbb{C}[\lambda,\lambda^{-1}]$ is the algebra of Laurent polynomials in the complex parameter λ. g' may be represented as the class of elements $\partial_{n,g}$ $(n \in \mathbb{Z},\ g \in \underset{\sim}{g})$, equipped with the bracket

$$[\partial_{n_1,g_1}, \partial_{n_2,g_2}] = \partial_{n_1+n_2,[g_1,g_2]} \qquad (2.1)$$

Now one wishes to construct a realization of $\underset{\sim}{g}'$ as a family of differential operators acting on a space F of functions (dynamical fields). The elements $\partial_{n,g}$ will be regarded as defining differentiation (evolution) with respect to the parameters $t_{n,g}$. Such a realization will be called "dynamical".

If F is equipped with a Poisson bracket $\{\ ,\ \}$ then one may define the Hamiltonians $H_{n,g} \in F$ by

$$\partial_{n,g} u = \{u, H_{n,g}\} \qquad \forall u \in F \qquad (2.2)$$

Then the Jacobi identity implies that

$$\{u,\{H_{n_1,g_2}, H_{n_2,g_2}\}\} = [\partial_{n_1,g_1}, \partial_{n_2,g_2}]u$$

$$= \partial_{n_1+n_2,[g_1+g_2]}u$$

$$= \{u, H_{n_1+n_2,[g_1,g_2]}\} \qquad \forall u \in F \quad (2.3)$$

and so

$$\{H_{n_1,g_1}, H_{n_2,g_2}\} = H_{n_1+n_2,[g_1,g_2]} + n_1 \delta_{n_1,-n_2} \delta_{g_1,g_2} c \quad (2.4)$$

where $c \in \mathbb{C}$ is a constant. (The constant term has been written in its most general form [6], which can always be arrived at by a suitable redefinition of the Hamiltonians, since they are only defined up to addition of constants.) This means that for any dynamical realization there is an associated "Hamiltonian

realization" of the algebra (with central extension, if $c \neq 0$).

For some $p \in \mathbb{Z}$, $E \in \underset{\sim}{g}$, let $t_{p,E}$ be a distinguished parameter which will be called the "space dimension", labelled x. Without loss of generality, it will be assumed that p is positive and E belongs to a Cartan subalgebra $\underset{\sim}{h}$ of $\underset{\sim}{g}$. Then $\underset{\sim}{g}$ may be decomposed as

$$\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m} \qquad (2.5)$$

where $\underset{\sim}{k}$ is the centralizer of E (see Appendix). The collection

$$\underline{T} = \{t_{n,k} : n \in \mathbb{Z},\ k \in \underset{\sim}{k}\} \qquad (2.6)$$

will be called the "evolution space". Any element of $\underline{T}$ may be singled out and regarded as the "time dimension", labelled t. Then the manifold spanned by (x,t) will be called the "physical space", and the operator ∂_t will define the equation of motion for any function $u(x,t) \in F$. The whole class $\partial_{n,k}$ (where $t_{n,k} \in \underline{T}$) defines a collection of equations of motion, which will be referred to as the "dynamical equations". The equations defined by the operators $\partial_{n,E}$ ($n \in \mathbb{Z}$) will be called the "central hierarchy", while those defined by $\partial_{N,E}$ ($N \geq 0$) will be called the "fundamental hierarchy".

The collection

$$\underline{I} = \{t_{n,m} : n \in \mathbb{Z},\ m \in \underset{\sim}{m}\} \qquad (2.7)$$

consists of parameters whose evolution operators $\partial_{n,m}$ do not commute with the spatial translation operator ∂_x (i.e. momentum is not conserved for the evolution equations). $\underline{I}$ will be called the "internal" space, and the operators $\partial_{n,m}$ will be regarded as defining translations in the "internal" (non-physical) dimensions $t_{n,m}$.

For all the dynamical equations, momentum $(H_{p,E})$ will be conserved. In particular, this means that

$$0 = \partial_{-p,E} H_{p,E} = \{H_{p,E}, H_{-p,E}\} = pc \tag{2.8}$$

(from (2.4)), and so the central term actually vanishes (if $p \neq 0$).

Next notice that (2.4) implies

$$\{H_{n,k}\ ,\ H_{m,E}\} = 0 \qquad \forall\, n,m \in \mathbf{Z},\ k \in \underset{\sim}{k} \tag{2.9}$$

i.e. the Hamiltonians $H_{n,k}$ are conserved quantities for the equations of the central hierarchy. The subset consisting of $H_{n,h}$ $(n \in \mathbf{Z},\ h \in \underset{\sim}{h})$ forms a maximal involutive class of conserved quantities. Furthermore, for any step operator $e_\alpha \in \underset{\sim}{m}$, one may define

$$\Gamma_\alpha \equiv \Sigma_{n=-\infty}^{\infty} \lambda^{-n} H_{n,e_\alpha} \tag{2.10}$$

(where λ is a formal parameter). Then, using (2.4) (and the notation $[E, e_\alpha] = (\alpha . E) e_\alpha$),

$$\begin{aligned} \partial_{n,E} \Gamma_\alpha &= \Sigma_{m=-\infty}^{\infty} \lambda^{-m} \{H_{m,e_\alpha}\ ,\ H_{n,E}\} \\ &= -\Sigma_{m=-\infty}^{\infty} \lambda^{-m} (\alpha . E) H_{m+n,e_\alpha} \\ &= -\lambda^{n} (\alpha . E) \Gamma_\alpha \qquad\qquad \forall\, n \in \mathbf{Z},\ e_\alpha \in \underset{\sim}{m} \end{aligned} \tag{2.11}$$

and so the equations of motion are linearized. Eqs.(2.9),(2.11) show that the equations of the central hierarchy are completely integrable. For the equations defined by $\partial_{n,h}$ $(n \in \mathbf{Z},\ h \in \underset{\sim}{h},\ h \neq E)$, there may be fewer conserved quantities (i.e. only those of the maximal involutive class), but the linearization (2.11) still applies. For the equations defined by $\partial_{n,e}$ $(n \in \mathbf{Z},\ e \in \underset{\sim}{k}, e \notin \underset{\sim}{h})$, there are still infinitely many conserved quantities $(H_{m,e}: m \in \mathbf{Z})$, but the equations are not linearized by (2.10).

In order to construct a dynamical realization, one begins by considering the equation

$$\partial_{n_1,g_1} A_{n_2,g_2} - \partial_{n_2,g_2} A_{n_1,g_1} + [A_{n_1,g_1}, A_{n_2,g_2}] = A_{n_1+n_2,[g_1,g_2]} \tag{2.12}$$

for an unknown class of "potentials" $A_{n,g} \in F \otimes \mathbf{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. (Notice that when $[g_1, g_2] = 0$, Eq.(2.12) becomes a zero curvature condition).

One seeks solutions $A_{n,g}$ for (2.12). The simplest example is the constant solution

$$A_{n,g} = \lambda^{n} g \tag{2.13}$$

Now consider the "gauge transformation"

$$A_{n,g} \to \gamma A_{n,g} \gamma^{-1} - \gamma_{n,g} \gamma^{-1} \equiv A_{n,g}^{(\gamma)} \tag{2.14}$$

where γ belongs to the loop group, with coefficients in $\mathbf{F}$ (and $\gamma_{n,g} \equiv \partial_{n,g} \gamma$). Making use of the identities

$$\partial_{n,g}(\gamma^{-1}) = -\gamma^{-1}\gamma_{n,g}\gamma^{-1} \qquad (2.15a)$$

$$\partial_{n,g}(\gamma A\gamma^{-1}) = [\gamma_{n,g}\gamma^{-1}, \gamma A\gamma^{-1}] + \gamma\partial_{n,g}A\gamma^{-1} \quad \forall A\in g \qquad (2.15b)$$

Eq.(2.12) is transformed to

$$\partial_{n_1,g_1}A^{(\gamma)}_{n_2,g_2} - \partial_{n_2,g_2}A^{(\gamma)}_{n_1,g_1} + [A^{(\gamma)}_{n_1,g_1}, A^{(\gamma)}_{n_2,g_2}] = A^{(\gamma)}_{n_1+n_2,[g_1,g_2]} \qquad (2.16)$$

i.e. the equation (2.12) is form-invariant under (2.14). Therefore, using (2.13) in (2.14), a solution is

$$A_{n,g} = \lambda^n\gamma g\gamma^{-1} - \gamma_{n,g}\gamma^{-1} \qquad (2.17)$$

where γ is arbitrary. Now choose two possible forms:

$$\gamma_1 = \phi\omega = \phi\exp\Sigma^{\infty}_{n=1}\lambda^{-n}\omega_n \qquad (2.18a)$$

$$\gamma_2 = \Psi\Omega = \Psi\exp\Sigma^{\infty}_{n=1}\lambda^{n}\Omega_n \qquad (2.18b)$$

where ϕ,Ψ are independent of λ and $\omega_n,\Omega_n \in F\otimes g$. With these two choices, $A_{n,g}$ will be an infinite series in either descending or ascending powers.

Now impose the condition that $A_{n,g}$ is the same for either choice of γ. Then $A_{n,g}$ must be a polynomial of degree n in only positive (negative) powers of λ if n is positive (negative). Let N be a positive integer. Then (using (2.15a))

$$A_{N,g} = \lambda^N\phi\omega g\omega^{-1}\phi^{-1} - \phi\omega_{N,g}\omega^{-1}\phi^{-1} - \phi_{N,g}\phi^{-1} \qquad (2.19a)$$

$$= \lambda^N\Psi\Omega g\Omega^{-1}\Psi^{-1} - \Psi\Omega_{N,g}\Omega^{-1}\Psi^{-1} - \Psi_{N,g}\Psi^{-1} \qquad (2.19b)$$

$$\equiv \Sigma^N_{n=0}\lambda^n A^n_{N,g} \qquad (2.19c)$$

Equating coefficients of powers of λ, one obtains

$$(\omega_{N,g}\omega^{-1})_n = (\omega g\omega^{-1})_{N+n} \qquad \forall\, n>0 \qquad (2.20a)$$

$$A_{N,g} = \Sigma^N_{n=0}\lambda^n\phi(\omega g\omega^{-1})_{N-n}\phi^{-1} - \phi_{N,g}\phi^{-1} \qquad (2.20b)$$

$$(\Omega_{N,g}\Omega^{-1})_n = (\Omega g\Omega^{-1})_{n-N} - \Psi^{-1}A^n_{N,g}\Psi \qquad \forall\, n>0 \qquad (2.20c)$$

$$\Psi_{N,g}\Psi^{-1} = \phi_{N,g}\phi^{-1} - \phi(\omega g\omega^{-1})_N\phi^{-1} \qquad \forall\, N>0 \qquad (2.20d)$$

$$\Psi_{0,g}\Psi^{-1} = \phi_{0,g}\phi^{-1} - \phi g\phi^{-1} + \Psi g\Psi^{-1} \qquad (2.20e)$$

where $(f(\omega))_n$ denotes the coefficient of λ^{-n} (=0 if n<0) and $(f(\Omega))_n$ denotes the coefficient of λ^n (=0 if n<0). Repeating the process with -N in (2.19), one obtains

$$(\Omega_{-N,g}\Omega^{-1})_n = (\Omega g\Omega^{-1})_{N+n} \qquad \forall\, n>0 \qquad (2.21a)$$

$$A_{-N,g} = \Sigma^N_{n=0}\lambda^{-n}\Psi(\Omega g\Omega^{-1})_{N-n}\Psi^{-1} - \Psi_{-N,g}\Psi^{-1} \qquad (2.21b)$$

$$(\omega_{-N,g}\omega^{-1})_n = (\omega g\omega^{-1})_{n-N} - \phi^{-1}A^n_{-N,g}\phi \qquad \forall\, n>0 \qquad (2.21c)$$

$$\Psi_{-N,g}\Psi^{-1} = \phi_{-N,g}\phi^{-1} + \Psi(\Omega g\Omega^{-1})_N\Psi^{-1} \qquad \forall\, N>0 \qquad (2.21d)$$

where $A^{n}_{-N,g}$ denotes the coefficient of λ^{-n} (=0 if n>N).

Now write $x \equiv t_{p,E}$. From Eq.(2.20), one has

$$(\omega_x\omega^{-1})_n = (\omega E\omega^{-1})_{p+n} \qquad \forall\, n>0 \qquad (2.22a)$$

$$A_x = \Sigma^{p}_{n=0}\lambda^{n}\phi(\omega E\omega^{-1})_{p-n}\phi^{-1} - \phi_x\phi^{-1} \qquad (2.22b)$$

$$(\Omega_x\Omega^{-1})_n = (\Omega E\Omega^{-1})_{n-p} - \Psi^{-1}A^{n}_{x}\Psi \qquad \forall\, n>0 \qquad (2.22c)$$

Eq.(2.22a) defines the generators ω_n as functions of x. The coefficients $\omega_1,\dots,\omega_{p-1};\omega_p^{\underset{\sim}{m}}$ (where the superscript is used to denote the $\underset{\sim}{m}$ component), which appear in A_x, are not uniquely determined by (2.22a), and will be called the "dynamical sector". It will be seen that the dynamical sector may be chosen so that ω_n is determined to all orders by (2.22a). ϕ is undetermined, and may be chosen arbitrarily. Ψ, also, is not explicitly determined, but is fixed by (2.20d), (2.20e), (2.21d). The generators Ω_n are determined by (2.22c) in terms of Ψ and the dynamical sector.

The evolution of ϕ is subject to the consistency condition

$$\partial_{n_1,g_1}(\phi^{-1}\phi_{n_2,g_2}) - \partial_{n_1,g_1}(\phi^{-1}\phi_{n_2,g_2})$$

$$+ [\phi^{-1}\phi_{n_1,g_1},\ \phi^{-1}\phi_{n_2,g_2}] = \phi^{-1}\phi_{n_1+n_2,[g_1,g_2]} \qquad (2.23)$$

i.e. $\phi^{-1}\phi_{n,g}$ is a λ-independent solution of Eq.(2.12). If ϕ is chosen to be the identity, then (from (2.20d,e),(2.21d)) Ψ is fixed by

$$\Psi_{N,g}\Psi^{-1} = -(\omega g\omega^{-1})_N \qquad \forall\, N>0 \qquad (2.24a)$$

$$\Psi_{0,g}\Psi^{-1} = \Psi g\Psi^{-1} \qquad (2.24b)$$

$$\Psi_{-N,g}\Psi^{-1} = \Psi(\Omega g\Omega^{-1})_N\Psi^{-1} \qquad \forall\, N>0 \qquad (2.24c)$$

This case will be called the "basic gauge". Now, since $\phi^{-1}\phi_{n,g}$ is a solution of Eq.(2.12), one can choose

$$\phi^{-1}\phi_{n,g} = A^{(\phi)0}_{n,g} \qquad \forall\, n\in\mathbb{Z},\ g\in\mathbf{g} \qquad (2.25)$$

where $A^{(\phi)}_{n,g}$ is a solution in any gauge ϕ; i.e. (from (2.20b), (2.21b,d) with ϕ replaced by ϕ)

$$\phi^{-1}\phi_{N,g} = \phi(\omega g\omega^{-1})_N\phi^{-1} - \phi_{N,g}\phi^{-1} \qquad (2.26a)$$

$$\phi^{-1}\phi_{-N,g} = -\phi_{-N,g}\phi^{-1} \qquad \forall\, N>0 \qquad (2.26b)$$

Letting ϕ be the basic gauge (i.e. ϕ=1), one obtains the "principal gauge":

$$\phi^{-1}\phi_{N,g} = (\omega g\omega^{-1})_N \qquad \forall\, N\geq 0 \qquad (2.27a)$$

$$\phi^{-1}\phi_{-N,g} = 0 \qquad \forall\, N>0 \qquad (2.27b)$$

In this gauge, Eqs.(2.20d,e),(2.21d) become

$$\Psi^{-1}\Psi_{N,g} = 0 \qquad \forall N>0 \tag{2.28a}$$

$$\Psi^{-1}\Psi_{-N,g} = (\Omega g\Omega^{-1})_N \qquad \forall N\geq 0 \tag{2.28b}$$

(Letting ϕ be the principal gauge in (2.26), one returns to the basic gauge).

One may also fix ϕ by choosing $\phi_x\phi^{-1}$ as a function of the dynamical sector. Then the dynamical equations for ϕ may be written as

$$\phi^{-1}\phi_{n,k} = \partial^{-1}(\phi^{-1}\partial_{n,k}(\phi_x\phi^{-1})\phi) \tag{2.29}$$

which satisfies Eq.(2.23) for $g\in \underset{\sim}{k}$. (In general, one cannot extend the non-local definition (2.29) to the internal translations $\phi^{-1}\phi_{n,m}$ ($m\in \underset{\sim}{m}$). Eq.(2.23) may no longer hold, since ∂_x does not commute with these translations). Gauges for which $\phi^{-1}\phi_{N,E}$ (the fundamental hierarchy) is local will be called "local gauges" (although ϕ itself is a non-local gauge transformation).

3. THE GAUGE EQUATIONS

It will now be shown how ω and Ω are determined from the "gauge equations" (2.22a),(2.22c)

$$(\omega_x\omega^{-1})_n = (\omega E\omega^{-1})_{p+n} \tag{3.1}$$

$$(\Omega_x\Omega^{-1})_n = (\Omega E\Omega^{-1})_{n-p} - \Psi^{-1}\phi(\omega E\omega^{-1})_{p-n}\phi^{-1}\Psi \tag{3.2}$$

for all $n>0$. Recalling the definitions (2.18), one can expand in powers of λ to obtain the identities

$$(\omega_\mu\omega^{-1})_n = \Sigma^n_{r=1}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\dots[\omega_{k_{r-1}}, \partial_\mu\omega_{k_r}]\dots]] \tag{3.3a}$$

$$(\omega A\omega^{-1})_n = \Sigma^n_{r=1}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\dots[\omega_{k_r}, A]\dots]] \qquad \forall A\in g \tag{3.3b}$$

(and similarly for Ω). If $p=0$, then (3.1),(3.2) become (using (3.3))

$$\partial_x\omega_n = [\omega_n, E] \tag{3.4a}$$

$$\partial_x\Omega_n = [\Omega_n, E] \tag{3.4b}$$

i.e. one can choose $\omega_n = \Omega_n = 0$ for all n. Henceforth, it will be assumed that $p\geq 1$.

Putting n=1 in (3.2), one obtains (using (3.3) again)

$$\partial_x\Omega_1 = -\Psi^{-1}\phi(\omega E\omega^{-1})_{p-1}\phi^{-1}\Psi \qquad \forall p>1 \tag{3.5a}$$

$$\partial_x\Omega_1 = E - \Psi^{-1}\phi E\phi^{-1}\Psi \qquad p=1 \tag{3.5b}$$

and so Ω_1 is determined (non-locally) in terms of ω,ϕ,Ψ. Now suppose that all $\Omega_{j<n}$ are determined. Then (3.2) gives

$$\partial_x\Omega_n + \Sigma_1(\text{terms in } \Omega_{j<n}) = \Sigma_2(\text{terms in } \Omega_{j<n}) - \Psi^{-1}\phi(\omega E\omega^{-1})_{p-n}\phi^{-1}\Psi \tag{3.6}$$

and so Ω_n is determined to all orders in terms of ω, ϕ, Ψ.

For the gauge equation (3.1), the situation is more complicated. First, let p=1. Then, putting n=1 in Eq.(3.1)

$$\partial_x \omega_1 = [\omega_2, E] + 1/2[\omega_1[\omega_1, E]] \tag{3.7}$$

This may be split into $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components (using (A.5)):

$$\partial_x \omega_1^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}} \tag{3.8a}$$

$$[\omega_2, E] = \partial_x \omega_1^{\underset{\sim}{m}} - 1/2[\omega_1^{\underset{\sim}{k}}[\omega_1, E]] - 1/2[\omega_1^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{m}} \tag{3.8b}$$

Eq.(3.8a) determines $\omega_1^{\underset{\sim}{k}}$ (non-locally) in terms of the dynamical sector $\omega_1^{\underset{\sim}{m}}$. Then $\omega_2^{\underset{\sim}{m}}$ is determined by Eq.(3.8b). Now suppose that $\omega_1, \cdots, \omega_{n-1}; \omega_n^{\underset{\sim}{m}}$ are determined. Then, using (3.3), Eq.(3.1) (with p=1) becomes

$$\partial_x \omega_n + \Sigma_1 = [\omega_{n+1}, E] + 1/2[\omega_n[\omega_1, E]] + 1/2[\omega_1[\omega_n, E]] + \Sigma_2 \tag{3.9}$$

(where Σ_j are sums of commutators involving $\omega_{m<n}$). Then the $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components are

$$\partial_x \omega_n^{\underset{\sim}{k}} = 1/2[\omega_n^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}} + 1/2[\omega_1^{\underset{\sim}{m}}[\omega_n, E]]^{\underset{\sim}{k}} + \Sigma_3 \tag{3.10a}$$

$$[\omega_{n+1}, E] = \partial_x \omega_n^{\underset{\sim}{m}} - 1/2[\omega_n[\omega_1, E]]^{\underset{\sim}{m}} - 1/2[\omega_1[\omega_n, E]]^{\underset{\sim}{m}} + \Sigma_4 \tag{3.10b}$$

Eq.(3.10a) determines $\omega_n^{\underset{\sim}{k}}$ non-locally. Then $\omega_{n+1}^{\underset{\sim}{m}}$ is determined by Eq.(3.10b). So ω_n is determined to all orders.

Now consider the case p=2. Putting n=1 in Eq.(3.1), the $\underset{\sim}{k}$ component is

$$\partial_x \omega_1^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_2, E]]^{\underset{\sim}{k}} + 1/2[\omega_2^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}} + 1/6[\omega_1[\omega_1[\omega_1, E]]]^{\underset{\sim}{k}} \tag{3.11}$$

One is free to choose the dynamical sector $(\omega_1, \omega_2^{\underset{\sim}{m}})$, subject to Eq.(3.11). (One such choice will be discussed later on). Now suppose that $\omega_{n<N}; \omega_N^{\underset{\sim}{m}}$ are determined in terms of the dynamical sector. Then the $\underset{\sim}{m}$ component of Eq.(3.1) with p=2, n=N-1 takes the form

$$[\omega_{N+1}, E] + 1/2[\omega_N^{\underset{\sim}{k}}[\omega_1, E]] = \Sigma_1 \tag{3.12}$$

(where Σ_j involves the coefficients $\omega_{n<N}; \omega_N^{\underset{\sim}{m}}$). Now put n=N in Eq.(3.1) (with p=2). The $\underset{\sim}{k}$ component gives

$$\partial_x \omega_N^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_{N+1}, E]]^{\underset{\sim}{k}} + 1/2[\omega_{N+1}^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}}$$

$$+ 1/6[\omega_N^{\underset{\sim}{k}}[\omega_1^{\underset{\sim}{m}}[\omega_1, E]]]^{\underset{\sim}{k}} + 1/6[\omega_1^{\underset{\sim}{m}}[\omega_N^{\underset{\sim}{k}}[\omega_1, E]]]^{\underset{\sim}{k}} + \Sigma_2 \tag{3.13}$$

Substituting (3.12) and using (A.8), this can be rewritten

$$\partial_x \omega_N^{\underset{\sim}{k}} = -1/3[\omega_1^{\underset{\sim}{m}}[\omega_N^{\underset{\sim}{k}}[\omega_1, E]]]^{\underset{\sim}{k}} + 1/6[\omega_N^{\underset{\sim}{k}}[\omega_1^{\underset{\sim}{m}}[\omega_1, E]]]^{\underset{\sim}{k}} + \Sigma_3 \tag{3.14}$$

$$= \Sigma_3 \tag{3.15}$$

using (A.10). This determines $\omega_N^{\underset{\sim}{k}}$, and then $\omega_{N+1}^{\underset{\sim}{m}}$ is determined by (3.12). So ω_n is determined to all orders.

For $p>2$, the sort of cancellation which appears in (3.14) will not occur, and one must choose the dynamical sector so that ω_n may be determined to all orders in terms of $\omega_{j<n}$. Such a choice will not be unique. For example, for any $p>2$ let q be a positive integer such that $2q\leq p-1$, and write

$$q = 2r + \sigma \tag{3.16}$$

where $\sigma \in \{0,1\}$. Now choose

$$\omega_1,\ldots\omega_{p-q-1} = 0 \tag{3.17a}$$

$$\omega^{\underset{\sim}{m}}_{p-q},\ldots,\omega^{\underset{\sim}{m}}_{p-r-1} = 0 \tag{3.17b}$$

One must check the consistency of (3.17). For $1\leq n\leq p$ the gauge equation (3.1) becomes (using (3.3))

$$\partial_x\omega_n = [\omega_{n+p},E] + 1/2\Sigma^{n+q}_{j=p-r}[\omega_{n+p-j}[\omega_j,E]] \tag{3.18}$$

For $1\leq n\leq p-q-r-1$ this reduces to

$$[\omega_{n+p},E] = 0 \tag{3.19}$$

i.e. $\omega^{\underset{\sim}{m}}_{p+1},\ldots,\omega^{\underset{\sim}{m}}_{2p-q-r-1} = 0$. Then, for $p-q-r\leq n\leq p-q-1$, Eq.(3.18) gives

$$[\omega_{n+p},E] = -1/2\Sigma^{n+q}_{j=p-r}[\omega_{n+p-j}[\omega_j,E]] \tag{3.20}$$

The $\underset{\sim}{k}$ component of this equation is zero, by (3.17b). So

$\omega^{\underset{\sim}{m}}_{2p-q-r},\ldots,\omega^{\underset{\sim}{m}}_{2p-q-1}$ are now known in terms of $\omega_{p-q},\ldots,\omega_{p-1}$.

For $p-q\leq n\leq p-r-1$, the $\underset{\sim}{k}$ component of Eq.(3.18) is

$$\partial_x\omega_n = 1/2\Sigma^{n+r}_{j=p-r}[\omega^{\underset{\sim}{m}}_{n+p-j}[\omega_j,E]]^{\underset{\sim}{k}} \tag{3.21}$$

(using (3.17)). This determines $\omega_{p-q},\ldots,\omega_{p-r-1}$ in terms of $\omega^{\underset{\sim}{m}}_{p-r},\ldots,\omega^{\underset{\sim}{m}}_{p-1}$. The $\underset{\sim}{m}$ component is

$$[\omega_{n+p},E] = -1/2\Sigma^{n+q}_{j=p-r}[\omega^{\underset{\sim}{k}}_{n+p-j}[\omega_j,E]] - 1/2\Sigma^{n+r}_{j=p-r}[\omega^{\underset{\sim}{m}}_{n+p-j}[\omega_j,E]]^{\underset{\sim}{m}} \tag{3.22}$$

For $p-r\leq n\leq p$ the $\underset{\sim}{k}$ component of (3.18) becomes (using (3.17),(3.19))

$$\partial_x\omega^{\underset{\sim}{k}}_n = 1/2\Sigma^{n}_{j=p-r}[\omega^{\underset{\sim}{m}}_{n+p-j}[\omega_j,E]]^{\underset{\sim}{k}} \tag{3.23}$$

and so $\omega^{\underset{\sim}{k}}_{p-r},\ldots,\omega^{\underset{\sim}{k}}_{p}$ are determined. Then $\omega^{\underset{\sim}{m}}_{2p-q},\ldots,\omega^{\underset{\sim}{m}}_{2p-r-1}$ are determined by (3.22).

So far, $\omega_1,\ldots,\omega_p;\omega^{\underset{\sim}{m}}_{p+1},\ldots,\omega^{\underset{\sim}{m}}_{2p-r-1}$ have been consistently determined in terms of $\omega^{\underset{\sim}{m}}_{p-r},\ldots,\omega^{\underset{\sim}{m}}_{p}$ (which may be regarded as the dynamical fields of the system). Now suppose that $\omega_{j<N}$ and $\omega^{\underset{\sim}{m}}_{N},\ldots,\omega^{\underset{\sim}{m}}_{N+p-r-2}$ have been determined (where $N\geq p+1$). The $\underset{\sim}{k}$ component of the gauge equation (3.1) (with $n=N$) may be written as

$$\partial_x\omega^{\underset{\sim}{k}}_N = 1/2\Sigma^{N+q}_{j=p-q}[\omega^{\underset{\sim}{m}}_j[\omega_{N+p-j},E]]^{\underset{\sim}{k}} + \Sigma(\text{terms in } \omega_{j<N}) \tag{3.24}$$

and so $\omega^{\underset{\sim}{k}}_N$ is determined. The $\underset{\sim}{m}$ component of (3.1) with $n=N-r-1$ has the form

$$[\omega_{N+p-r-1},E] = -1/2\Sigma_{j=p-q}^{N-1}[\omega_j[\omega_{N+p-r-1-j},E]]^{\underset{\sim}{m}} + \Sigma(\text{terms in } \omega_{j<N}) \tag{3.25}$$

and so $\omega^{\underset{\sim}{m}}_{N+p-r-1}$ is determined. Hence ω_n is determined to all orders. So for any p, it is possible to choose the dynamical sector so that ω may be determined.

The gauge equation (3.1) determines $\omega^{\underset{\sim}{k}}_n$ non-locally, and so one would expect the equations of motion (2.20) to be non-local. This is true in general, but for systems in which the dynamical sector is local, the fundamental hierarchy are in fact local equations of motion (in any local gauge). To show this, let w be of the form

$$w = \exp\Sigma_{n=1}^{\infty}\lambda^{-n}w_n \tag{3.26}$$

and define the transformed gauge potentials

$$a_{n,g} = w^{-1}A_{n,g}w + w^{-1}w_{n,g} \tag{3.27}$$

Now choose

$$w_n = \omega_n \qquad 1\leq n\leq p-1 \tag{3.28a}$$

$$w^{\underset{\sim}{m}}_p = \omega^{\underset{\sim}{m}}_p \tag{3.28b}$$

i.e. ω,w coincide on the dynamical sector. Then (recalling (2.22b), and letting ϕ be the basic gauge), a_x takes the form

$$a_x = \lambda^p E + \Sigma_{n=1}^{\infty}\lambda^{-n}a_x^{-n} \tag{3.29}$$

while the transformed potentials of the fundamental hierarchy are of the form

$$a_{N,E} = \lambda^N E + \Sigma_{n=-\infty}^{N-p-1}\lambda^n a^n_{N,E} \tag{3.30}$$

(using (2.20b) with ϕ=1,g=E). Now specify the remaining generators of w by choosing $a_x \in \underset{\sim}{k}$, i.e.

$$(w^{-1}Ew)^{\underset{\sim}{m}}_{n+p} = -(w^{-1}w_x)^{\underset{\sim}{m}}_n - \Sigma_{r=0}^{p-1}(w^{-1}A_x^r w)^{\underset{\sim}{m}}_{r+n} \qquad \forall n\geq 1 \tag{3.31}$$

and so

$$[w_{n+p},E] = \Sigma(\text{terms in } w_{j<n+p}) \tag{3.32}$$

Then $w^{\underset{\sim}{m}}_n$ (n>p) is locally defined to all orders in terms of the dynamical sector. $w^{\underset{\sim}{k}}_n$ (n$\geq$p) is left undetermined, and will be chosen to vanish.

In the gauge (3.27), the equations of motion of the fundamental hierarchy have the representation

$$\partial_x a_{N,E} - \partial_{N,E}a_x + [a_x,a_{N,E}] = 0 \tag{3.33}$$

(from Eq.(2.12)). It will now be shown that $a_{N,E}$ may actually be written as

$$a_{N,E} = \lambda^N E + \Sigma_{n=1}^{\infty}\lambda^{-n}a^{-n}_{N,E} \tag{3.34}$$

This is certainly the case if N$\leq$p (from (3.30)). Suppose N>p.

Substituting (3.29),(3.30), one may write the coefficient of λ^{N-n} in (3.33) as

$$\partial_x a^{N-n}_{N,E} + [E,\ a^{N-p-n}_{N,E}] + \Sigma^{n-p-1}_{r=1}[a^{-r}_x,\ a^{N-n+r}_{N,E}] = 0 \qquad 1\leq n\leq N \tag{3.35}$$

i.e.

$$[E,\ a^{N-p-n}_{N,E}] = 0 \qquad 1\leq n\leq p \tag{3.36a}$$

$$\partial_x a^{N-n}_{N,E} + [E,\ a^{N-p-n}_{N,E}] + \Sigma^{n-p-1}_{r=1}[a^{-r}_x,\ a^{N-n+r}_{N,E}] = 0 \qquad p+1\leq n\leq N \tag{3.36b}$$

Eq.(3.36a) shows that $a^{N-2p}_{N,E},\ldots,a^{N-p-1}_{N,E} \in \underset{\sim}{k}$. Then the $\underset{\sim}{m}$ component of Eq.(3.36b) becomes

$$[E, a^{N-p-n}_{N,E}] = -\partial_x (a^{N-n}_{N,E})^{\underset{\sim}{m}} - \Sigma^{n-2p-1}_{r=1}[a^{-r}_x, (a^{N-n+r}_{N,E})^{\underset{\sim}{m}}] \qquad p+1\leq n\leq N \tag{3.37}$$

i.e.

$$[E,\ a^{N-p-n}_{N,E}] = 0 \qquad p+1\leq n\leq \min(N,2p) \tag{3.38}$$

and so the upper bound on the summation in (3.37) is reduced to n-3p-1. Repeating until min(N,mp) = N for some m, one finds that

$$a^{n}_{N,E} \in \underset{\sim}{k} \qquad -p\leq n\leq N-p-1 \tag{3.39}$$

Then the $\underset{\sim}{k}$ component of Eq.(3.36b) is

$$\partial_x a^{N-n}_{N,E} = -\Sigma^{n-p-1}_{r=1}[a^{-r}_x,\ a^{N-n+r}_{N,E}] \qquad p+1\leq n\leq N \tag{3.40}$$

Putting n=p+1, one has

$$\partial_x a^{N-p-1}_{N,E} = 0 \tag{3.41}$$

so one may choose $a^{N-p-1}_{N,E} = 0$. Then, by induction, Eq.(3.40) allows one to choose

$$a^{N-n}_{N,E} = 0 \qquad p+1\leq n\leq N \tag{3.42}$$

and so the assertion (3.34) is proved. Now invert the gauge transformation (3.27) to obtain

$$A_{N,E} = wa_{N,E}w^{-1} - w_{N,E}w^{-1}$$

$$=\lambda^{N} wEw^{-1} + \Sigma^{\infty}_{n=1}\lambda^{-n} wa^{-n}_{N,E}w^{-1} - w_{N,E}w^{-1} \tag{3.43}$$

Since $A_{N,E}$ is a polynomial in only positive powers of λ, one may equate coefficients to get

$$A_{N,E} = \Sigma^{N}_{n=0}\lambda^{n}(wEw^{-1})_{N-n} \tag{3.44a}$$

$$= \Sigma^{N}_{n=0}\lambda^{n}(\omega E\omega^{-1})_{N-n} \qquad \forall\ N\geq 0 \tag{3.44b}$$

(from (2.20b) with ψ=1), and so

$$wEw^{-1} = \omega E\omega^{-1} \tag{3.45}$$

(where the constants of integration in $\omega E\omega^{-1}$ are set to zero). The equations of motion of the fundamental hierarchy may be represented as

$$\partial_x A_{N,E} - \partial_{N,E} A_x + [A_x, A_{N,E}] = 0 \tag{3.46}$$

(from Eq.(2.12)). If the dynamical sector is local, then w (and hence $A_{N,E}$ (3.44a)) will be local, and so the equations of motion (3.46) are local. The associated Hamiltonian realization will consist of non-local Hamiltonians with a sub-class of local quantities $H_{N,E}$.

4. REALIZATIONS WITH p = 1

In the basic gauge, with p=1, the gauge potential (2.22b) is

$$A_x = \lambda E + [\omega_1, E] \tag{4.1}$$

For G=SU(2), this defines the AKNS system [7]. Since the Cartan subalgebra is one-dimensional, the equations of motion all belong to the central hierarchy.

Now, from (2.21c) with N=n=1 and $\phi=1$,

$$\partial_{-1,E}\tilde{\omega}_1^m = (-A^1_{-1,E})^{\tilde{m}}$$

$$= (-\Psi E \Psi^{-1})^{\tilde{m}} \tag{4.2}$$

(using (2.21b). In the basic gauge, one also has ((2.24a) with N=1,g=E)

$$\Psi_x \Psi^{-1} = -[\omega_1, E] \tag{4.3}$$

so that the equation of motion may be rewritten (with $\partial_t \equiv \partial_{-1,E}$) as

$$\partial_t(\Psi_x \Psi^{-1}) = [\Psi E \Psi^{-1}, E] \tag{4.4}$$

For G=SU(2), choose

$$E = 1/\sqrt{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.5}$$

$$[\omega_1, E] = \begin{pmatrix}0 & u\\ -u & 0\end{pmatrix} \tag{4.6}$$

where u is a real field. Then (noting (4.3)), Ψ may be written as

$$\Psi = \begin{pmatrix}\exp(i\beta)\cos(\alpha/2) & \exp(-i\beta)\sin(\alpha/2)\\ -\exp(i\beta)\sin(\alpha/2) & \exp(-i\beta)\cos(\alpha/2)\end{pmatrix} \tag{4.7}$$

where α_x=2u and β_x=0. Then Eq.(4.4) becomes

$$\alpha_{xt} = \sin\alpha \tag{4.8}$$

which is the sine-Gordon equation.

Now consider the equation of motion with $\partial_t \equiv \partial_{2,E}$. Using (2.20a) one has

$$\partial_t \omega_1 = (\omega_{2,E}\omega^{-1})_1 = (\omega E \omega^{-1})_3 \tag{4.9}$$

and, since $\partial_x \equiv \partial_{1,E}$, Eq.(2.20a) gives

$$(\omega E\omega^{-1})_3 = (\omega_x\omega^{-1})_2$$

$$= \partial_x\omega_2 + 1/2[\omega_1, \partial_x\omega_1] \qquad (4.10)$$

(from Eq.(3.3a)). Then

$$\partial_t\overset{m}{\underset{\sim}{\omega}}_1 = \partial_x\overset{m}{\underset{\sim}{\omega}}_2 + 1/2[\overset{k}{\underset{\sim}{\omega}}_1, \partial_x\overset{m}{\underset{\sim}{\omega}}_1] + 1/2[\overset{m}{\underset{\sim}{\omega}}_1, \partial_x\overset{k}{\underset{\sim}{\omega}}_1] + 1/2[\overset{m}{\underset{\sim}{\omega}}_1, \partial_x\overset{m}{\underset{\sim}{\omega}}_1]^{\overset{m}{\sim}} \qquad (4.11)$$

This may be evaluated using (3.8). One obtains

$$\partial_t[\omega_1,E] = \partial_{xx}\overset{m}{\underset{\sim}{\omega}}_1 - [\partial_x\overset{m}{\underset{\sim}{\omega}}_1[\omega_1,E]] - 1/2[E[\overset{m}{\underset{\sim}{\omega}}_1[\overset{m}{\underset{\sim}{\omega}}_1[\omega_1,E]]]] \qquad (4.12)$$

For G=SU(2), one may choose

$$E = -1/2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad (4.13a)$$

$$[\omega_1,E] = \begin{pmatrix}0 & q\\ r & 0\end{pmatrix} \qquad (4.13b)$$

If g is restricted to the compact or non-compact real form ($r=\pm q^*$) then Eq.(4.12) becomes

$$iq_t = q_{xx} \mp 2q|q|^2 \qquad (4.14)$$

which is the non-linear Schrodinger equation. For a symmetric Lie algebra $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$ the middle term of Eq.(4.12) will vanish, and the equation of motion (when $\underset{\sim}{g}$ is restricted to the compact or non-compact real form) is the matrix form of the generalized

non-linear Schrödinger equation discussed in [4]. The Hamiltonian realization associated with this system was constructed in [2,3].

Now consider the gauge choice $\phi \in K$ (the Lie group of $\underset{\sim}{k}$). Then (2.22b) with p=1 is

$$A_x = \lambda E + [\phi\omega_1\phi^{-1}, E] - \phi_x\phi^{-1} \qquad (4.15)$$

For G=SU(2), choose

$$E = -1/2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad (4.16a)$$

$$[\phi\omega_1\phi^{-1}, E] = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad (4.16b)$$

$$\phi_x\phi^{-1} = \begin{pmatrix}u & 0\\ 0 & -u\end{pmatrix} \qquad (4.16c)$$

For $\partial_t \equiv \partial_{2,E}$ write

$$\phi_t\phi^{-1} = \begin{pmatrix}v & 0\\ 0 & -v\end{pmatrix} \qquad (4.17)$$

The consistency condition (2.23) on ϕ becomes

$$u_t = v_x \qquad (4.18)$$

and one also has (using (2.15b))

$$0 = \partial_t[\phi\omega_1\phi^{-1}, E]$$

$$= [[\phi_t\phi^{-1}, \phi\omega_1\phi^{-1}]E] + [\phi\partial_t\omega_1\phi^{-1}, E] \qquad (4.19)$$

$$= [\phi_t\phi^{-1}[\phi\omega_1\phi^{-1},E]] - [\partial_x(\phi_x\phi^{-1}),\ \phi\overset{m}{\tilde{\omega}}_1\phi^{-1}]$$

$$+ [\phi_x\phi^{-1}[\phi_x\phi^{-1},\ \phi\overset{m}{\tilde{\omega}}_1\phi\ \ ]] \qquad (4.20)$$

(using (A.7),(4.12),(4.16b), and using (2.15b) to rewrite $\phi\partial_{xx}\overset{m}{\tilde{\omega}}_1\phi^{-1}$). Substituting (4.16),(4.17), this becomes

$$v = u_x + u^2 \qquad (4.21)$$

Then the equation of motion is given by (4.18)

$$u_t = u_{xx} + 2uu_x \qquad (4.22)$$

which is the Burgers equation. ϕ plays the rôle of the Cole-Hopf transformation [8,9].

In the principal gauge, with p=1, the gauge potential is

$$A_x = \lambda\phi E\phi^{-1} \qquad (4.23)$$

(from (2.22b),(2.27a)). The equation of motion with $\partial_t \equiv \partial_{2,E}$ is

$$\partial_t(\phi E\phi^{-1}) = [\phi_{2,E}\phi^{-1},\ \phi E\phi^{-1}] \qquad \text{(by (2.15b))}$$

$$= \phi[(\omega E\omega^{-1})_2,E]\phi^{-1} \qquad \text{(by (2.27a))}$$

$$= \phi[\partial_x\omega_1,E]\phi^{-1} \qquad \text{(by (2.22a))}$$

$$= \phi\partial_x(\phi^{-1}\phi_x)\phi^{-1} \qquad \text{(by (2.27a))}$$

$$= \partial_x(\phi_x\phi^{-1}) \qquad \text{(by (2.15a))} \qquad (4.24)$$

Now let $g = \underset{\sim}{k} \oplus \underset{\sim}{m}$ be a symmetric algebra and choose E so that $(\mathrm{ad}E)^2 = -I$ on $\underset{\sim}{m}$. Then (since $\phi^{-1}\phi_x \in \underset{\sim}{m}$, by (2.27a)), Eq.(4.24) may be rewritten

$$\partial_t(\phi E\phi^{-1}) = -\partial_x(\phi[[\phi^{-1}\phi_x,\ E],\ E]\phi^{-1})$$

$$= \partial_x[\phi E\phi^{-1},\ \partial_x(\phi E\phi^{-1})] \qquad (4.25)$$

Writing $\phi E\phi^{-1} = S$, this becomes

$$\partial_t S = [S,\ S_{xx}] \qquad (4.26)$$

which is the generalized Heisenberg ferromagnet equation [4]. The gauge equivalence of the non-linear Schrodinger and Heisenberg ferromagnet equations was first demonstrated in [10].

5. REALIZATIONS WITH p = 2

It was shown in Section 3 (Eq.(3.11)) that for p=2 the dynamical sector must satisfy

$$\partial_x\overset{k}{\tilde{\omega}}_1 = 1/2[\overset{m}{\tilde{\omega}}_1[\omega_2,E]]^{\overset{k}{\sim}} + 1/2[\overset{m}{\tilde{\omega}}_2[\omega_1,E]]^{\overset{k}{\sim}} + 1/6[\omega_1[\omega_1[\omega_1,E]]]^{\overset{k}{\sim}} \quad (5.1)$$

Let $g = \underset{\sim}{k} \oplus \underset{\sim}{m}$ be a symmetric algebra with $(\mathrm{ad}E)^2 = -I$ on $\underset{\sim}{m}$. Then one may choose

$$\overset{k}{\tilde{\omega}}_1 = 0 \qquad (5.2a)$$

$$\overset{m}{\tilde{\omega}}_2 = 0 \qquad (5.2b)$$

Now one may consistently choose

$$\omega_n \in \underset{\sim}{m} \qquad (n \text{ odd}) \tag{5.3a}$$

$$\omega_n \in \underset{\sim}{k} \qquad (n \text{ even}) \tag{5.3b}$$

To see why this is so, note that the commutation relations

$$[\underset{\sim}{k}, \underset{\sim}{k}] \subset \underset{\sim}{k} \tag{5.4a}$$

$$[\underset{\sim}{k}, \underset{\sim}{m}] \subset \underset{\sim}{m} \tag{5.4b}$$

$$[\underset{\sim}{m}, \underset{\sim}{m}] \subset \underset{\sim}{k} \tag{5.4c}$$

may be realized by $\underset{\sim}{k} \to \{2n\}$, $\underset{\sim}{m} \to \{2n+1\}$ $(n \in \mathbb{Z})$ and $[a,b] \to a+b$. Then by induction on n in (3.3) one obtains

$$(\omega_\mu \omega^{-1})_n \in \begin{cases} \underset{\sim}{m} & (n \text{ odd}) \\ \underset{\sim}{k} & (n \text{ even}) \end{cases} \tag{5.5a}$$

$$(\omega k \omega^{-1})_n \in \begin{cases} \underset{\sim}{m} & (n \text{ odd}) \\ \underset{\sim}{k} & (n \text{ even}) \end{cases} \qquad \forall k \in \underset{\sim}{k} \tag{5.5b}$$

$$(\omega m \omega^{-1})_n \in \begin{cases} \underset{\sim}{k} & (n \text{ odd}) \\ \underset{\sim}{m} & (n \text{ even}) \end{cases} \qquad \forall m \in \underset{\sim}{m} \tag{5.5c}$$

and so the gauge equation

$$(\omega_x \omega^{-1})_n = (\omega E \omega^{-1})_{n+2} \tag{5.6}$$

is consistent. Then (5.5) and (2.20a) imply that

$$\partial_{N,k}\omega = 0 \qquad (N \geq 0 \text{ odd},\ k \in \underset{\sim}{k}) \tag{5.7a}$$

$$\partial_{N,m}\omega = 0 \qquad (N \geq 0 \text{ even},\ m \in \underset{\sim}{m}) \tag{5.7b}$$

Now let $\phi \in K$. Then the gauge potential (2.22b), with the choice (5.2), becomes

$$A_x = \lambda^2 E + \lambda[\phi\omega_1\phi^{-1}, E] + 1/2[\phi\omega_1\phi^{-1}[\phi\omega_1\phi^{-1}, E]] - \phi_x\phi^{-1} \tag{5.8}$$

Now consider the equation of motion with $\partial_t \equiv \partial_{4,E}$. From (2.20a)

$$\partial_t\omega_1 = (\omega E \omega^{-1})_5 = (\omega_x\omega^{-1})_3 \qquad \text{(using (2.22a))}$$

$$= \partial_x\omega_3 + 1/2[\omega_2, \partial_x\omega_1] + 1/2[\omega_1, \partial_x\omega_2]$$

$$+ 1/6[\omega_1[\omega_1, \partial_x\omega_1]] \qquad \text{(from (3.3a))} \tag{5.9}$$

To obtain an explicit expression, ω_2 and ω_3 must be determined from the gauge equation (5.6). Putting n=1, one obtains

$$[\omega_3, E] = \partial_x\omega_1 - 1/2[\omega_2[\omega_1, E]] - 1/6[\omega_1[\omega_1[\omega_1, E]]] \tag{5.10}$$

while n=2 gives (using (A.8))

$$\partial_x\omega_2 = [\omega_1[\omega_3, E]] + 1/6[\omega_2[\omega_1[\omega_1, E]]] + 1/6[\omega_1[\omega_2[\omega_1, E]]]$$

$$+ 1/24[\omega_1[\omega_1[\omega_1[\omega_1, E]]]] - 1/2[\omega_1, \partial_x\omega_1] \tag{5.11}$$

Substituting (5.10) and using (A.10) this becomes

$$\partial_x\omega_2 = 1/2[\omega_1,\ \partial_x\omega_1] - 1/8[\omega_1[\omega_1[\omega_1,E]]] \tag{5.12}$$

Then the equation of motion (5.9) is

$$\partial_t\omega_1 = [E,\ \partial_{xx}\omega_1] + 2/3[\omega_1[\omega_1,\ \partial_x\omega_1]] - 1/6\partial_x[E[\omega_1[\omega_1,E]$$

$$- 1/8[\omega_1[\omega_1[\omega_1[\omega_1,E]]]]] \tag{5.13}$$

Now, (using (2.15b), (A.7))

$$\partial_t A^1_x = \partial_t[\phi\omega_1\phi^{-1},\ E]$$

$$= [\phi_t\phi^{-1}[\phi\omega_1\phi^{-1},\ E]] + [\phi(\partial_t\omega_1)\phi^{-1},\ E] \tag{5.14}$$

Substituting (5.13), and using (2.15b) to rewrite $\phi(\partial^n_x\omega_1)\phi^{-1}$, this becomes

$$\partial_t A^1_x = [\phi_t\phi^{-1},\ A^1_x] + [E,\ \partial_{xx}A^1_x] + [\partial_x(\phi_x\phi^{-1})[A^1_x,E]]$$

$$+ 2[\phi_x\phi^{-1}[\partial_x A^1_x,E]] - [\phi_x\phi^{-1}[\phi_x\phi^{-1}[A^1_x,E]]]$$

$$+ 2/3[A^1_x[A^1_x,\ \partial_x A^1_x]] - 1/6\partial_x[E[A^1_x[A^1_x[A^1_x,E]]]]$$

$$+ 1/6[\phi_x\phi^{-1}[E[A^1_x[A^1_x[A^1_x,E]]]]] + 2/3[A^1_x[A^1_x[A^1_x,\ \phi_x\phi^{-1}]]]$$

$$- 1/8[A^1_x[A^1_x[A^1_x[A^1_x[A^1_x,E]]]]] \tag{5.15}$$

$$\equiv [\phi_t\phi^{-1},\ A^1_x] + F \tag{5.16}$$

Now choose

$$\phi_x\phi^{-1} = \alpha[A^1_x[A^1_x,E]] \tag{5.17}$$

Then $\phi_t\phi^{-1}$ is defined by the consistency condition

$$\partial_x(\phi_t\phi^{-1}) = \partial_t(\phi_x\phi^{-1}) + [\phi_x\phi^{-1},\ \phi_t\phi^{-1}] \tag{5.18}$$

which may be rewritten using (5.16) (and (A.7,8,10))

$$\partial_x(\phi_t\phi^{-1}) = 2\alpha[\partial_t A^1_x[A^1_x,E]] - \alpha[\phi_t\phi^{-1}[A^1_x[A^1_x,E]]$$

$$= 2\alpha[F[A^1_x,E]] \tag{5.19}$$

This becomes (using (A.7,8,12,13,16))

$$\partial_x(\phi_t\phi^{-1}) = 2\alpha[A^1_x,\ \partial_{xx}A^1_x] + (4\alpha^2+\alpha)[A^1_x[\partial_x A^1_x[A^1_x[A^1_x,E]]]]$$

$$+ (4\alpha^2-2\alpha)[A^1_x[A^1_x[\partial_x A^1_x[A^1_x,E]]]] \tag{5.20}$$

i.e. (using (A.15))

$$\phi_t\phi^{-1} = 2\alpha[A^1_x,\ \partial_x A^1_x] + (\alpha^2 + \alpha/4)[A^1_x[A^1_x[A^1_x[A^1_x,E]]]]$$

$$+ (4\alpha^2 - 2\alpha)\partial^{-1}[A^1_x[A^1_x[\partial_x A^1_x[A^1_x,E]]]] \tag{5.21}$$

Substituting this and (5.17) into (5.15) gives the equation of motion

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] + (2\alpha - 4\alpha^2)[A_x^1, \partial^{-1}[A_x^1[A_x^1[\partial_x A_x^1[A_x^1, E]]]]]$$

$$+ 1/2[E[\partial_x A_x^1[A_x^1[A_x^1, E]]]] + (4\alpha - 1)[E[A_x^1[\partial_x A_x^1[A_x^1, E]]]]$$

$$+ (\alpha/4 - 1/8)[A_x^1[A_x^1[A_x^1[A_x^1[A_x^1, E]]]]] \qquad (5.22)$$

(where (A.12,13,16) have been used). This equation becomes local for two values of α. Putting α=0 (the basic gauge) gives a quintic non-linear Schrodinger equation. Putting α = 1/2 (the principal gauge), one obtains

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] + 1/2\partial_x[E[A_x^1[A_x^1[A_x^1, E]]]] \qquad (5.23)$$

When $\mathbf{g}$ is restricted to the compact or non-compact real form, the components of (5.23) give the derivative non-linear Schrodinger equation of Fordy [11].

For $\mathbf{g}$ = sl(2,$\mathbb{C}$), one can write

$$A_x^1 = qe_+ + re_- \qquad (5.24)$$

and one can check that for this case

$$[A_x^1[A_x^1[\partial_x A_x^1[A_x^1, E]]]] = 4\partial_x[A_x^1[A_x^1[A_x^1[A_x^1, E]]]] \qquad (5.25)$$

so $\phi_t\phi^{-1}$ (5.20) is local for all α. The equation of motion (5.22) is then

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] - (\alpha - 1/2)(\alpha - 1/4)[A_x^1[A_x^1[A_x^1[A_x^1[A_x^1, E]]]]]$$

$$+ 1/2[E[\partial_x A_x^1[A_x^1[A_x^1, E]]]] + (4\alpha - 1)[E[A_x^1[\partial_x A_x^1[A_x^1, E]]]] \qquad (5.26)$$

Putting α = 1/4, this becomes

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] + 1/2[E[\partial_x A_x^1[A_x^1[A_x^1, E]]]] \qquad (5.27)$$

The restriction $r = \pm q^*$ gives the derivative non-linear Schrodinger equation of Chen et al [12].

The gauge potentials of the zero curvature representation are easily found. Substituting (5.17) into (5.8) gives

$$A_x = \lambda^2 E + \lambda A_x^1 + (1/2 - \alpha)[A_x^1[A_x^1, E]] \qquad (5.28)$$

while from (2.20b)

$$A_t = \lambda^4 E + \lambda^3[\phi\omega_1\phi^{-1}, E] + \lambda^2[\phi\omega_1\phi^{-1}[\phi\omega_1\phi^{-1}, E]]$$

$$+\lambda\phi(\partial_x\omega_1)\phi^{-1} + \phi(\partial_x\omega_2 + 1/2[\omega_1, \partial_x\omega_1])\phi^{-1} - \phi_t\phi^{-1} \qquad (5.29)$$

(where (5.6) has been used). This becomes (using (5.11),(5.17), (5.21))

$$A_t = \lambda^4 E + \lambda^3 A_x^1 + \lambda^2[A_x^1[A_x^1, E]]$$

$$+ \lambda([E, \partial_x A_x^1] + \alpha[E[A_x^1[A_x^1[A_x^1, E]]]])$$

$$+ (1-2\alpha)[A_x^1, \partial_x A_x^1] - (\alpha^2 - 3\alpha/4 + 1/8)[A_x^1[A_x^1[A_x^1[A_x^1, E]]]]$$

$$- (4\alpha^2 - 2\alpha)\partial^{-1}[A_x^1[A_x^1[\partial_x A_x^1[A_x^1, E]]]] \qquad (5.30)$$

6. DISCUSSION

Various authors have used gauge invariance to investigate the conserved quantities of integrable dynamical systems, such as the nonlinear sigma model in [13] (where the gauge potentials are rational functions of λ), and the Toda equation in [14]. The gauge transformation w (3.31) is a generalization of the local transformation used in [14]. Non-local conserved quantities were constructed for the non-linear sigma model in [15], and these are associated with infinitesimal transformations which realize the "positive half" of a centre-free Kac-Moody algebra [16]. However, the charges themselves do not form an algebra [17]. The infinitesimal symmetries of the SU(2) non-linear Schrödinger equation were investigated in [18], using the gauge transformation to the Heisenberg ferromagnet. Only the "positive" subalgebra was realized non-trivially, since the gauge transformation is in fact the principal gauge (4.23), which is trivial in the negative part of the algebra (2.27b).

Much work has been done by the Kyoto group [19] on the construction of soliton solutions for non-linear dynamical systems using vertex operators. The construction of solitons within the formalism of this paper, and the connection with the work of the Japanese authors, will be dealt with in a subsequent paper.

Another topic which will be investigated further is the construction of the Hamiltonian realizations. One must first define the Poisson bracket between matrix elements of the dynamical sector (with different values of the spectral parameter), i.e.

$$\{A_x(x_1,\lambda_1) \underset{,}{\otimes} A_x(x_2,\lambda_2)\} \tag{6.1}$$

(where the notation of [20] is used). Such objects have been studied in connection with the "r-matrix" [21] and Yang-Baxter equation [22].

APPENDIX

Let $\underset{\sim}{g}$ be a semisimple Lie algebra over a field P of characteristic zero. (By "semisimple" it is meant that $\underset{\sim}{g}$ has a symmetric bilinear form (,) which is non-degenerate). A Cartan subalgebra of $\underset{\sim}{g}$ is a maximal abelian subalgebra $\underset{\sim}{h}$ with the property that, for all $h \in \underset{\sim}{h}$, adh is a semisimple endomorphism of $\underset{\sim}{g}$. (Recall that $(\mathrm{ad}h)g \equiv [h,g]$, and a semisimple endomorphism is one for which the roots of the minimal polynomial are all distinct). Every semisimple Lie algebra has at least one Cartan subalgebra. Choose one, and let E be an element of it. Define

$$\underset{\sim}{k} = \{g \in \underset{\sim}{g} : [E,g] = 0\} \tag{A.1a}$$

$$\underset{\sim}{l} = \{g \in \underset{\sim}{g} : (\mathrm{ad}E)^n g = 0\} \tag{A.1b}$$

$$\underset{\sim}{m} = [E,\underset{\sim}{g}] \tag{A.1c}$$

Being semisimple, adE is diagonalizable over P (or its algebraic closure), and hence $\underset{\sim}{k} = \underset{\sim}{l}$. Then $\underset{\sim}{k} \cap \underset{\sim}{m} = 0$. Also, the fact that $\underset{\sim}{g}$ is semisimple, together with the relation $0 = (\underset{\sim}{g},[\underset{\sim}{k},E]) = ([E,\underset{\sim}{g}],\underset{\sim}{k})$, implies that $\dim \underset{\sim}{k} + \dim \underset{\sim}{m} = \dim \underset{\sim}{g}$; i.e. $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$ (direct sum as vector spaces). If there exist $k \in \underset{\sim}{k}$, $m \in \underset{\sim}{m}$ such that $[k,m] \in \underset{\sim}{k}$ $(\neq 0)$, then (writing $m = [E,m']$)

$$0 = [E[k[E,m']]] = [E[E[k,m']]] \tag{A.2}$$

(by the Jacobi identity) and so

$$0 = (\underset{\sim}{g},[E[E[k,m']]]) = ([\underset{\sim}{g},E],[E[k,m']]) \tag{A.3}$$

However,

$$(\underset{\sim}{k},[E[k,m']]) = ([\underset{\sim}{k},E],[k,m']) = 0 \tag{A.4}$$

i.e. $[E[k,m']] = [k,m]$ is orthogonal to $\underset{\sim}{g}$, which contradicts the fact that $\underset{\sim}{g}$ is semisimple. So one has the relations

$$[\underset{\sim}{k},\underset{\sim}{m}] \subset \underset{\sim}{m} \qquad [\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \tag{A.5}$$

(the latter follows from the Jacobi identity on $[E[\underset{\sim}{k},\underset{\sim}{k}]]$). The corresponding Lie group G/K is called a reductive homogeneous space. For certain algebras it is possible to choose E so that one has the additional relation

$$[\underset{\sim}{m},\underset{\sim}{m}] \subset \underset{\sim}{k} \tag{A.6}$$

then G/K is called a symmetric space, and $\underset{\sim}{g}$ is called a symmetric Lie algebra. An important example arises when adE is a complex structure on $\underset{\sim}{m}$ (i.e. $(\mathrm{adE})^2 = -I$ on $\underset{\sim}{m}$). For further details, see [23].

In the general (reductive) case, it is clear (using the Jacobi identity) that

$$[k[m,E]] = [[k, m],E] \qquad \forall k \in \underset{\sim}{k},\ m \in \underset{\sim}{m} \tag{A.7}$$

$$[m_1[m_2,E]]^{\underset{\sim}{k}} = [m_2[m_1,E]]^{\underset{\sim}{k}} \qquad \forall m_1,m_2 \in \underset{\sim}{m} \tag{A.8}$$

(where the superscript indicates the $\underset{\sim}{k}$ component). Also, for all $k \in \underset{\sim}{k}$, $m \in \underset{\sim}{m}$,

$$\begin{aligned}[k[m[m,E]]] &= [m[k[m,E]]] - [[m,E],[k,m]] \\ &= 2[m[k[m,E]]] + [E[m[k,m]]]\end{aligned} \tag{A.9}$$

i.e.

$$[k[m[m,E]]]^{\underset{\sim}{k}} = 2[m[k[m,E]]]^{\underset{\sim}{k}} = 2[[E,m],[k,m]]^{\underset{\sim}{k}} \tag{A.10}$$

From now on it is assumed that E can be (and is) chosen so that $(\mathrm{adE})^2 = \kappa^2$ (constant) on $\underset{\sim}{m}$ (i.e. the symmetric case). Notice that

$$[[E,m_1],[E,m_2]] = -\kappa^2[m_1,m_2] \qquad \forall m_1,m_2 \in \underset{\sim}{m} \tag{A.11}$$

Also

$$\begin{aligned}[m[E[m[m[m,E]]]]] &= [m[[m[m,E]],[m,E]]] && \text{(by (A.7))} \\ &= 1/2[[m[m,E]],[m[m,E]]] && \text{(by (A.10))} \\ &= 0\end{aligned} \tag{A.12}$$

and

$$[m[E[m[m[m[m[m,E]]]]]]]$$

$$= [m[[E,m],[m[m[m[m,E]]]]]] \qquad \text{(by (A.7))}$$

$$= [m[[[E,m]m],[m[m[m,E]]]]] + [m[m[[E,m],[m[m[m,E]]]]]]$$

(by the Jacobi identity)

$$= [[m[m,E]],[m[m[m[m,E]]]]]$$

(by the Jacobi identity and (A.8),(A.12))

$$= [m[[m,E],[m[m[m[m,E]]]]]] - [[m,E],[m[m[m[m,E]]]]]]$$

(by the Jacobi identity)

$$= -2[m[E[m[m[m[m[m,E]]]]]]] \qquad \text{(by (A.7),(A.8))}$$

$$= 0 \qquad \text{(A.13)}$$

Now let $m,m' \in \underset{\sim}{m}$. Then the Jacobi identity implies

$$[m'[m[m[m,E]]]]$$

$$= [[m',m],[m[m,E]]] + [m[m'[m[m,E]]]]$$

$$= 2[m[[m',m],[m,E]]]] + [m[m'[m[m,E]]]] \qquad \text{(by (A.10))}$$

$$= 3[m[m'[m[m,E]]]] - 2[m[m[m'[m,E]]]] \qquad \text{(A.14)}$$

In particular, if ∂ is a differential operator then (using (A.8))

$$\partial[m[m[m[m,E]]]] = 4[m[\partial m[m[m,E]]]] \qquad \text{(A.15)}$$

Lastly, since $(\mathrm{ad}E)^2 = \kappa^2$ on $\underset{\sim}{m}$, one has for all $m,m' \in \underset{\sim}{m}$:

$$[m[m,m']] = \kappa^{-2}[E[E[m[m,m']]]]$$

$$= \kappa^{-2}[E[[E,m],[m,m']]] \qquad \text{(by (A.7))}$$

$$= \kappa^{-2}[E[m[m'[m,E]]]] - \kappa^{-2}[E[m'[m[m,E]]]] \qquad \text{(A.16)}$$

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