

Kac-Moody Symmetry of Generalized Non-Linear Schrödinger Equations

A. D. W. B. Crumey

Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, United Kingdom

Abstract. The classical non-linear Schrödinger equation associated with a symmetric Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is known to possess a class of conserved quantities which form a realization of the algebra $\mathfrak{k} \otimes \mathbb{C}[\lambda]$. The construction is now extended to provide a realization of the Kac-Moody algebra $\mathfrak{k} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ (with central extension). One can then define auxiliary quantities to obtain the full algebra $\mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. This leads to the formal linearization of the system.

1. Introduction

This is a continuation of the work presented in [1], in which it was shown how to construct conserved quantities for the generalized non-linear Schrödinger (GNLS) equation of Fordy and Kulish [2]:

$$iq_t^\alpha = q_{xx}^\alpha \pm q^\beta q^\gamma q^{\delta*} R_{\beta\gamma-\delta}^\alpha \quad (1.1)$$

(summation is implied over repeated indices) which is associated with a Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. $q(x, t)$ is a matrix field in $1 + 1$ dimensions whose components lie in $\mathfrak{m}$, and $\mathfrak{k}$ is the centralizer of a special Cartan subalgebra element E satisfying the property

$$[E, e_\alpha] = -ie_\alpha \quad (1.2)$$

for all $e_\alpha \in \mathfrak{m}$ (where α is positive). This means that the algebra $\mathfrak{g}$ is “symmetric”, i.e.

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \quad (1.3)$$

The curvature tensor R has components in $\mathfrak{m}$ defined by

$$e_\alpha R_{\beta\gamma-\delta}^\alpha = [e_\beta[e_\gamma, e_{-\delta}]]. \quad (1.4)$$

Equation (1.1) can be written as a zero-curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0, \quad (1.5)$$

where

$$A_x = \lambda E + A_x^0, \quad (1.6a)$$

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]], \quad (1.6b)$$

and

$$A_x^0 = -q^\alpha e_\alpha - q^{-\alpha} e_{-\alpha}. \quad (1.7)$$

The component form of (1.5) is

$$iq_t^\alpha = q_{xx}^\alpha + q^\beta q^\gamma q^{-\delta} R_{\beta\gamma-\delta}^\alpha, \quad (1.8a)$$

$$-iq_t^{-\alpha} = q_{xx}^{-\alpha} + q^{-\beta} q^{-\gamma} q^\delta R_{-\beta-\gamma\delta}^{-\alpha}. \quad (1.8b)$$

The choices $q^{-\alpha} = \pm q^{\alpha*}$ correspond to the restriction to the non-compact (+) or compact (−) real forms of $\mathcal{G}$, and lead to Eq. (1.1) with a plus or minus sign.

One can find other values of A_t as a polynomial in λ such that the new equation of motion (1.5), with A_x given by (1.6a), is still independent of λ . Each such A_t is associated, via (1.5), with an evolution operator ∂_t . It was shown in [1] that when A_t is a polynomial in positive powers only, the collection of evolution operators can be labelled $\partial_{N,k}$, where $k \in \mathcal{K}$ and N is a positive integer, and that they have the commutation relation

$$[\partial_{M,j}, \partial_{N,k}] = \partial_{M+N,[j,k]} \quad \forall M, N \geq 0; \forall j, k \in \mathcal{K}. \quad (1.9)$$

In this paper, the case will be considered when A_t is a polynomial in negative powers of λ . This will lead to the construction of evolution operators $\partial_{-N,k}$ such that

$$[\partial_{m,j}, \partial_{n,k}] = \partial_{m+n,[j,k]} \quad \forall m, n \in \mathbb{Z}, \forall j, k \in \mathcal{K}. \quad (1.10)$$

The complete collection of operators $\partial_{\pm N,k}$ provides a realization of the Kac-Moody algebra $\mathcal{K} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, where $\mathbb{C}[\lambda, \lambda^{-1}]$ is the algebra of Laurent polynomials in the complex variable λ . The parameters $(\pm N, k)$ are thought of as infinitely many independent “time” variables.

In [1] it was shown how to construct a group element of the form,

$$\omega = \exp \sum_{n=1}^{\infty} \lambda^{-n} \omega_n, \quad (1.11)$$

defined by

$$\lambda \omega E \omega^{-1} - \omega_x \omega^{-1} = \lambda E + A_x^0. \quad (1.12)$$

Under the gauge transformation

$$A_x \rightarrow \omega^{-1} A_x \omega + \omega^{-1} \omega_x = \lambda E, \quad (1.13a)$$

$$A_t \rightarrow \omega^{-1} A_t \omega + \omega^{-1} \omega_t = a_t, \quad (1.13b)$$

where A_t is unknown, the zero curvature condition (1.5) becomes

$$\partial_x a_t + [\lambda E, a_t] = 0. \quad (1.14)$$

This equation can be satisfied by choosing

$$a_t = \lambda^N k, \quad (1.15)$$

where N is a positive integer and $k \in \mathcal{K}$ is constant. The transformation (1.13b) is

inverted to obtain

$$A_t = \lambda^N \omega k \omega^{-1} - \omega_t \omega^{-1}. \quad (1.16)$$

If A_t is chosen to have no negative powers of λ , then it is determined uniquely by (1.16) as the positive power part of $\lambda^N \omega k \omega^{-1}$, while the action of ∂_t on ω is determined by the negative power part. A_t and ∂_t defined in this way are denoted $A_N(k)$, $\partial_{N,k}$. Equating coefficients of powers of λ in (1.16) one obtains

$$A_N(k) = \sum_{n=0}^N \lambda^n (\omega k \omega^{-1})_{N-n}, \quad (1.17a)$$

$$(\omega_{N,k} \omega^{-1})_n = (\omega k \omega^{-1})_{N+n}, \quad (1.17b)$$

where $(\dots)_n$ denotes the coefficient of λ^{-n} . The relation (1.9) follows from the definition (1.17b).

Now suppose that one chooses

$$a_t = \lambda^{-N} k \quad (1.18)$$

as a solution to (1.14). Then the inverse gauge transformation (1.13b) gives

$$A_{-N}(k) = \lambda^{-N} \omega k \omega^{-1} - \omega_{-N,k} \omega^{-1} \quad (1.19)$$

which does not determine $A_{-N}(k)$. For example, if $N > 1$, then the coefficient of λ^{-1} in (1.19) is

$$A_{-N}^1(k) = -\partial_{-N,k} \omega_1. \quad (1.20)$$

One can think of (1.19) as defining the action of $\partial_{-N,k}$ on ω in terms of the as yet undetermined $A_{-N}(k)$. To find $A_{-N}(k)$ as a polynomial in negative powers of λ , one would like to have an equation like (1.19) in which ω is replaced by a group element which contains only non-negative powers of λ , i.e. one would like to find $\tilde{\omega}$ of the form

$$\tilde{\omega} = \exp \sum_{n=0}^{\infty} \lambda^n \tilde{\omega}_n \quad (1.21)$$

such that one can perform the transformation

$$A_x \rightarrow \tilde{\omega}^{-1} A_x \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_x = \lambda E, \quad (1.22a)$$

$$A_t \rightarrow \tilde{\omega}^{-1} A_t \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_t = a_t. \quad (1.22b)$$

Then one can again consider the solutions $a_t = \lambda^{\pm N} k$ for (1.14). For the case $\lambda^{-N} k$, the inverse transformation (1.22b) gives

$$A_{-N}(k) = \lambda^{-N} \tilde{\omega} k \tilde{\omega}^{-1} - \tilde{\omega}_{-N,k} \tilde{\omega}^{-1}, \quad (1.23)$$

which enables one to obtain $A_{-N}(k)$ and the action of $\partial_{-N,k}$ on $\tilde{\omega}$, by equating coefficients of powers of λ . The case $\lambda^N k$ defines the action of $\partial_{N,k}$ on $\tilde{\omega}$ in terms of $A_N(k)$ (1.17a). To construct $\tilde{\omega}$, one writes it in the form

$$\tilde{\omega} = \psi \Omega, \quad (1.24)$$

where ψ is independent of λ , and

$$\Omega = \exp \sum_{n=1}^{\infty} \lambda^n \Omega_n. \quad (1.25)$$

It will be shown in Sect. 2 that Eq. (1.22a) determines Ω to all orders in terms of the auxiliary field ψ . In Sect. 3 the commutation relations of the evolution operators $\partial_{\pm N, k}$ will be investigated, which will show them to form a realization of a Kac-Moody algebra. The class of operators can be extended by allowing the algebra element to be an arbitrary element of $\mathcal{g}$, rather than just of $\mathcal{k}$. In Sect. 4 the Hamiltonians for the operators $\partial_{\pm N, g}$ are considered. Their Poisson bracket algebra provides a realization of the Kac-Moody algebra $\mathcal{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathbb{C}c$. In Sect. 5 it is shown that the formal sum of Hamiltonians for the operators $\partial_{\pm N, e_a}$ can be used to linearize the system. The interpretation of these operators is discussed in Sect. 6.

2. Construction of $\tilde{\omega}$

Let $\tilde{\omega}$ be an element of the Lie group G , of the form

$$\tilde{\omega} = \psi \Omega, \quad (2.1)$$

where ψ is independent of λ , and

$$\Omega = \exp \sum_{n=1}^{\infty} \lambda^n \Omega_n. \quad (2.2)$$

Now fix $\tilde{\omega}$ by choosing

$$\lambda \tilde{\omega} E \tilde{\omega}^{-1} - \tilde{\omega}_x \tilde{\omega}^{-1} = A_x, \quad (2.3)$$

where A_x is given by (1.6a); i.e.

$$\lambda E + A_x^0 = \lambda \psi \Omega E \Omega^{-1} \psi^{-1} - \psi \Omega_x \Omega^{-1} \psi^{-1} - \psi_x \psi^{-1}. \quad (2.4)$$

Equating coefficients of powers of λ^0 , one can see that

$$A_x^0 = -\psi_x \psi^{-1}. \quad (2.5)$$

Notice that ψ is the group element which arises in the transformation between the GNLS system and the generalized Heisenberg ferromagnet [2]. (This will be explained more fully in Sect. 6.) The λ -dependent part of (2.4) becomes

$$\lambda \Omega E \Omega^{-1} - \Omega_x \Omega^{-1} = \lambda \psi^{-1} E \psi. \quad (2.6)$$

Now, by expanding (2.2) as a power series in λ , one can obtain the identities

$$(\Omega E \Omega^{-1})_n = \sum_{r=1}^n (r!)^{-1} \Sigma_{k_i: \Sigma k_i = n} [\Omega_{k_1} [\Omega_{k_2} [\cdots [\Omega_{k_r}, E] \cdots]]], \quad (2.7a)$$

$$(\Omega_x \Omega^{-1})_n = \sum_{r=1}^n (r!)^{-1} \Sigma_{k_i: \Sigma k_i = n} [\Omega_{k_1} [\Omega_{k_2} [\cdots [\Omega_{k_{r-1}}, \partial_x \Omega_{k_r}] \cdots]]], \quad (2.7b)$$

where $(\cdots)_n$ denotes the coefficient of λ^n . Use these to equate coefficients of λ^n in (2.6):

$$\lambda^1: \partial_x \Omega_1 = E - \psi^{-1} E \psi,$$

i.e

$$\Omega_1 = xE - \partial^{-1}(\psi^{-1} E \psi), \quad (2.8a)$$

$$\lambda^2: \partial_x \Omega_2 + 1/2[\Omega_1, \partial_x \Omega_1] = [\Omega_1, E],$$

i.e.

$$\Omega_2 = 1/2 \partial^{-1}([E, \partial^{-1}(\psi^{-1} E \psi)] + x[E, \psi^{-1} E \psi] + [\psi^{-1} E \psi, \partial^{-1}(\psi^{-1} E \psi)]), \quad (2.8b)$$

and so on. In general one has

$$(\Omega_x \Omega^{-1})_n = (\Omega E \Omega^{-1})_{n-1} \quad (2.9)$$

for all $n > 1$, and so $\partial_x \Omega_n$ is determined in terms of $\Omega_{m < n}$. In this way, Ω is determined to all orders non-locally in terms of ψ .

3. The Evolution Operators

Recall the zero curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0, \quad (3.1)$$

where A_x is given by (1.6a) and A_t is unknown. Consider the gauge transformation

$$A_x \rightarrow \omega^{-1} A_x \omega + \omega^{-1} \omega_x = \lambda E, \quad (3.2a)$$

$$A_t \rightarrow \omega^{-1} A_t \omega + \omega^{-1} \omega_t = a_t, \quad (3.2b)$$

where ω is the group element defined by (1.12), of the form $\omega = \exp \sum_{n=1}^{\infty} \lambda^{-n} \omega_n$. Under the transformation (3.2), the zero curvature condition (3.1) becomes

$$\partial_x a_t + [\lambda E, a_t] = 0. \quad (3.3)$$

One can choose the solutions

$$a_t = \lambda^{\pm N} k \quad (3.4)$$

for (3.3) (where N is a positive integer). Then (3.2b) can be inverted to obtain

$$A_N(k) = \lambda^N \omega k \omega^{-1} - \omega_{N,k} \omega^{-1}, \quad (3.5a)$$

$$\omega_{-N,k} \omega^{-1} = \lambda^{-N} \omega k \omega^{-1} - A_{-N}(k), \quad (3.5b)$$

where $A_N(k)$ is chosen to be a polynomial in non-negative powers of λ , while $A_{-N}(k)$ is a polynomial in negative powers. Equation (3.5a) defines $A_N(k)$ and the action of $\partial_{N,k}$ on ω , while (3.5b) is regarded as defining the action of $\partial_{-N,k}$ on ω in terms of $A_{-N}(k)$, which is still undetermined.

Now consider the gauge transformation (3.2) with ω replaced by $\tilde{\omega}$ as

constructed in Sect. 2. Then, from the definition (2.3),

$$A_x \rightarrow \tilde{\omega}^{-1} A_x \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_x = \lambda E, \quad (3.6a)$$

$$A_t \rightarrow \tilde{\omega}^{-1} A_t \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_t = a_t, \quad (3.6b)$$

where A_t is considered unknown. The zero curvature condition again takes the form (3.3), and the solutions (3.4) can be used to invert (3.6b) to give

$$A_{-N}(k) = \lambda^{-N} \tilde{\omega} k \tilde{\omega}^{-1} - \tilde{\omega}_{-N,k} \tilde{\omega}^{-1}, \quad (3.7a)$$

$$\tilde{\omega}_{N,k} \tilde{\omega}^{-1} = \lambda^N \tilde{\omega} k \tilde{\omega}^{-1} - A_N(k). \quad (3.7b)$$

Equation (3.7a) defines $A_{-N}(k)$ as the negative power part of $\lambda^{-N} \tilde{\omega} k \tilde{\omega}^{-1}$, and defines $\tilde{\omega}_{-N,k} \tilde{\omega}^{-1}$ as the positive power part. Equation (3.7b) defines the action of $\partial_{N,k}$ on $\tilde{\omega}$ in terms of $A_N(k)$, which was defined by the positive power part of (3.5a). Explicitly, one has

$$A_N(k) = \sum_{n=0}^N \lambda^n (\omega k \omega^{-1})_{N-n} = \sum_{n=0}^N \lambda^n A_N^n(k), \quad (3.8a)$$

$$(\omega_{N,k} \omega^{-1})_n = (\omega k \omega^{-1})_{N+n}, \quad (3.8b)$$

(from (3.5a)), where $(f(\omega))_n$ denotes the coefficient of λ^{-n} in $f(\omega)$,

$$A_{-N}(k) = \sum_{n=1}^N \lambda^{-n} \psi (\Omega k \Omega^{-1})_{N-n} \psi^{-1} = \sum_{n=1}^N \lambda^{-n} A_{-N}^n(k), \quad (3.9a)$$

$$(\Omega_{-N,k} \Omega^{-1})_n = (\Omega k \Omega^{-1})_{N+n}, \quad (3.9b)$$

for $N > 0$ (from (3.7a), using (2.1)), where $(f(\Omega))_n$ denotes the coefficient of λ^n in $f(\Omega)$,

$$\psi_{N,k} \psi^{-1} = -A_N^0(k) = -(\omega k \omega^{-1})_N \quad (\forall N > 0), \quad (3.10a)$$

$$\psi_{0,k} \psi^{-1} = \psi k \psi^{-1} - k \quad (3.10b)$$

(from the coefficient of λ^0 in (3.7b), using (2.1) and (3.8a)),

$$\psi_{-N,k} \psi^{-1} = \psi (\Omega k \Omega^{-1})_N \psi^{-1} \quad (\forall N > 0) \quad (3.11)$$

(from the coefficient of λ^0 in (3.7a)). Lastly, (3.5b) and (3.7b) become

$$\omega_{-N,k} \omega^{-1} = \lambda^{-N} \omega k \omega^{-1} - \sum_{n=1}^N \lambda^{-n} \psi (\Omega k \Omega^{-1})_{N-n} \psi^{-1} \quad (\forall N > 0), \quad (3.12a)$$

$$\Omega_{N,k} \Omega^{-1} = \lambda^N \Omega k \Omega^{-1} - \sum_{n=1}^N \lambda^n \psi^{-1} (\omega k \omega^{-1})_{N-n} \psi \quad (\forall N > 0), \quad (3.12b)$$

$$\Omega_{0,k} \Omega^{-1} = \Omega k \Omega^{-1} - k \quad (3.12c)$$

(using (3.9a), (3.8a) and (3.10)).

Notice that (3.8b) implies

$$(\omega_{1,E} \omega^{-1})_n = (\omega E \omega^{-1})_{n+1} = (\omega_x \omega^{-1})_n \quad (3.13)$$

(by (1.12)), i.e.

$$\partial_{1,E} = \partial_x \quad (3.14)$$

and so, by (3.10a),

$$\psi_x \psi^{-1} = -(\omega E \omega^{-1})_1 = -A_x^0 \quad (3.15)$$

(using (1.12) again). This is consistent with (2.5). Now, ω satisfies identities like (2.7), where $(\cdots)_n$ is taken to denote the coefficient of λ^{-n} , so that (from (3.15))

$$\begin{aligned} \partial_{N,k} A_x^0 &= [\partial_{N,k} \omega_1, E] = [(\omega_{N,k} \omega^{-1})_1, E] \quad (\text{using (2.7b)}) \\ &= [(\omega k \omega^{-1})_{N+1}, E] \quad (\text{from (3.8b)}), \end{aligned} \quad (3.16a)$$

and

$$\partial_{-N,k} A_x^0 = [\partial_{-N,k} \omega_1, E] = [E, A_{-N}^1(k)] \quad (\text{from the coefficient of } \lambda^{-1} \text{ in (3.5b)})$$

i.e.

$$-\partial_{-N,k}(\psi_x \psi^{-1}) = [E, \psi(\Omega k \Omega^{-1})_{N-1} \psi^{-1}] \quad (3.16b)$$

(by (3.9a) and (2.5)). Equations (3.16) are the equations of motion corresponding to $A_{\pm N}(k)$. (One could also obtain them by substitution of (3.8a), (3.9a) in (3.1).)

The commutation properties of the evolution operators will now be investigated. Recall Eq. (3.5):

$$\omega_{n,k} \omega^{-1} = \lambda^n \omega k \omega^{-1} - A_n(k) \quad \forall n \in \mathbb{Z}, k \in \ell \quad (3.17)$$

(or alternatively use $\tilde{\omega}$, i.e. Eq. (3.7)) and consider the identity

$$([\partial_{n,k}, \partial_{m,j}] \omega) \omega^{-1} = \partial_{n,k}(\omega_{m,j} \omega^{-1}) - \partial_{m,j}(\omega_{n,k} \omega^{-1}) + [\omega_{m,j} \omega^{-1}, \omega_{n,k} \omega^{-1}] \quad (3.18)$$

for all $m, n \in \mathbb{Z}, k, j \in \ell$. Using (3.17), one has

$$\partial_{n,k}(\omega_{m,j} \omega^{-1}) = \lambda^m [\lambda^n \omega k \omega^{-1} - A_n(k), \omega j \omega^{-1}] - \partial_{n,k} A_m(j). \quad (3.19)$$

One finds that (3.18) becomes

$$\begin{aligned} ([\partial_{n,k}, \partial_{m,j}] \omega) \omega^{-1} \\ = \partial_{m,j} A_n(k) - \partial_{n,k} A_m(j) - [A_n(k), A_m(j)] + \lambda^{n+m} \omega[k, j] \omega^{-1}, \end{aligned} \quad (3.20)$$

and so (using (3.17) to rewrite the last term in (3.20)) one can deduce that the relation

$$[\partial_{n,k}, \partial_{m,j}] = \partial_{n+m, [k, j]} \quad (3.21)$$

is equivalent to the condition

$$\partial_{n,k} A_m(j) - \partial_{m,j} A_n(k) + [A_n(k), A_m(j)] = A_{n+m}([k, j]), \quad (3.22)$$

which must now be verified using (3.8a) and (3.9a). One needs to consider separately the cases where n, m are of the same or different sign. Suppose they are of different sign, i.e. $n = N (\geq 0)$ and $m = -M (\leq 0)$, and suppose $(N - M) \geq 0$. Split (3.22) into coefficients of λ^{-p} and λ^q (where $p, q \geq 0$):

$$\partial_{N,k} A_{-M}^p(j) + [A_N(k), A_{-M}(j)]_{-p} = 0, \quad (3.23a)$$

$$-\partial_{-M,j} A_N^q(k) + [A_N(k), A_{-M}(j)]_q = A_{N-M}^q([k, j]). \quad (3.23b)$$

Using (3.8a) and (3.9a), the left side of Eq. (3.23a) becomes

$$\begin{aligned}
 & [\psi_{N,k}\psi^{-1}, \psi(\Omega j\Omega^{-1})_{M-p}\psi^{-1}] + \psi([\Omega_{N,k}\Omega^{-1}, \Omega j\Omega^{-1}]_{M-p})\psi^{-1} \\
 & + \sum_{r=0}^{M-p} [(\omega k\omega^{-1})_{N-r}, \psi(\Omega j\Omega^{-1})_{M-p-r}\psi^{-1}] \\
 & = \sum_{r=1}^{M-p} [(\omega k\omega^{-1})_{N-r}, \psi(\Omega j\Omega^{-1})_{M-p-r}\psi^{-1}] \\
 & + \sum_{r=1}^{M-p} \psi[(\Omega_{N,k}\Omega^{-1})_r, (\Omega j\Omega^{-1})_{M-p-r}]\psi^{-1} \quad (\text{using (3.10)}) \\
 & = 0
 \end{aligned}$$

as required (using (3.12b) and noting that $r < N$). The left side of (3.23b) becomes

$$\begin{aligned}
 & -([\omega_{-M,j}\omega^{-1}, \omega k\omega^{-1}]_{N-q} + \sum_{r=q+1}^{M+q} [(\omega k\omega^{-1})_{N-r}, A_{-M}^{-q}(j)]) \\
 & = (\omega[k, j]\omega^{-1})_{N-M-q} \quad (\text{using (3.9a), (3.12a)}) \\
 & = A_{N-M}^q([k, j]),
 \end{aligned}$$

as required (using (3.8a)).

The calculation for $(N - M) < 0$ proceeds along similar lines, noting that for this case $A_{N-M}([k, j])$ has the form (3.9a). One can also check the cases where n, m in (3.22) are of the same sign. (The case $n, m \geq 0$ was considered in [1]). Having verified (3.22), one can conclude, from (3.21), that the evolution operators $\partial_{\pm N,k}$ provide a realization of the Kac-Moody algebra $\mathfrak{k} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$.

Notice that (3.22) is a generalization of the zero curvature condition (3.1). Putting $(n, k) = (1, E)$ and recalling (3.14), Eq. (3.22) becomes

$$\partial_x A_m(j) - \partial_{m,j} A_x + [A_x, A_m(j)] = 0 \quad \forall m \in \mathbb{Z}, j \in \mathfrak{k}. \quad (3.24)$$

A less illuminating (but quicker) method of obtaining (3.21) is to use ψ in place of ω in (3.18). For example, using (3.10), (3.11) and Eq. (3.5), (3.7), one obtains

$$\begin{aligned}
 & ([\partial_{N,k}, \partial_{-M,j}]\psi)\psi^{-1} \\
 & = \sum_{r=1}^N [(\omega j\omega^{-1})_{r-M} - \psi(\Omega j\Omega^{-1})_{M-r}\psi^{-1}, (\omega k\omega^{-1})_{N-r}] \\
 & + \sum_{r=1}^M \psi[(\Omega k\Omega^{-1})_{r-N} - \psi^{-1}(\omega k\omega^{-1})_{N-r}\psi, (\Omega j\Omega^{-1})_{M-r}]\psi^{-1}. \quad (3.25)
 \end{aligned}$$

If $(N - M) \geq 0$ then (3.25) becomes

$$\begin{aligned}
 ([\partial_{N,k}, \partial_{-M,j}]\psi)\psi^{-1} & = \sum_{s=0}^{N-M} [(\omega j\omega^{-1})_{N-M-s}, (\omega k\omega^{-1})_s] \\
 & = -(\omega[k, j]\omega^{-1})_{N-M} = \psi_{N-M, [k, j]}\psi^{-1} \quad (3.26)
 \end{aligned}$$

with a similar result for $(N - M) < 0$. One can also check the equal sign cases in the same fashion.

The evolution operators $\partial_{\pm N,k}$ can be extended in the obvious way, simply by replacing k by a general constant element $g \in \mathcal{G}$, i.e.

$$A_n(g) = \lambda^n \omega g \omega^{-1} - \omega_{n,g} \omega^{-1} = \lambda^n \tilde{\omega} g \tilde{\omega}^{-1} - \tilde{\omega}_{n,g} \tilde{\omega}^{-1} \quad \forall n \in \mathbb{Z}, g \in \mathcal{G}. \quad (3.27)$$

One can then verify in the same way as earlier the equation

$$\partial_{n,g} A_m(h) - \partial_{m,h} A_n(g) + [A_n(g), A_m(h)] = A_{n+m}([g, h]) \quad \forall n, m \in \mathbb{Z}, g, h \in \mathcal{G}, \quad (3.28)$$

which is equivalent to the condition

$$[\partial_{n,g}, \partial_{m,h}] = \partial_{n+m,[g,h]}, \quad (3.29)$$

and so one obtains a realization of the Kac-Moody algebra $\mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. Notice that for $m \in \mathbb{Z}$ the quantities $A_{\pm N}(m)$ do not satisfy the zero curvature condition, which is why they were not obtained in the original construction (where one sought solutions of (3.1)), and so the parameters $(\pm N, m)$ cannot be regarded as true “times.” Their interpretation will be discussed later.

4. Hamiltonians

The Hamiltonians $H_n(g)$ for the operators $\partial_{n,g}$ are defined by

$$\partial_{n,g} f = \{f, H_n(g)\}, \quad (4.1)$$

where f is any function, and the Poisson bracket is given by

$$\begin{aligned} \{f_1, f_2\} = \sum_{\alpha} \int dz (\partial f_1 / \partial q^{-\alpha}(z) \cdot \partial f_2 / \partial q^{\alpha}(z) \\ - \partial f_1 / \partial q^{\alpha}(z) \cdot \partial f_2 / \partial q^{-\alpha}(z)). \end{aligned} \quad (4.2)$$

Hamilton's equations take the form

$$\partial_{n,g} q^{\alpha} = -\partial H_n(g) / \partial q^{-\alpha}, \quad (4.3a)$$

$$\partial_{n,g} q^{-\alpha} = \partial H_n(g) / \partial q^{\alpha}. \quad (4.3b)$$

Now, the Jacobi identity, together with (4.1) and (3.29), implies that (for any function f)

$$\begin{aligned} \{f\{H_n(g), H_m(h)\}\} &= [\partial_{n,g}, \partial_{m,h}] f = \partial_{n+m,[g,h]} f \\ &= \{f, H_{n+m}([g, h])\} \quad \forall n, m \in \mathbb{Z}, g, h \in \mathcal{G}, \end{aligned} \quad (4.4)$$

and so

$$\{H_n(g), H_m(h)\} = H_{n+m}([g, h]) + C_{n,m}^{g,h}, \quad (4.5)$$

where $C_{n,m}^{g,h}$ is a constant. Since the Hamiltonians are only defined up to addition of constants, it is possible to arrange for the Poisson bracket to take the form [3]

$$\{H_n(g), H_m(h)\} = H_{n+m}([g, h]) + n\delta_{n,-m}\delta_{g,h}c, \quad (4.6)$$

where c is a constant; i.e. the Hamiltonians provide a realization of the centrally extended algebra $\mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathbb{C}c$.

The Hamiltonian of the GNLS equation is $H_2(E)$, so Eq. (4.6) implies that $H_{\pm N}(k)$ are conserved quantities for the GNLS equation, except for $H_{-2}(E)$ if $c \neq 0$. Notice also that (4.6) and (4.1) imply

$$\partial_{\pm N, k} H_0(E) = 0 \quad \forall N \geq 0, k \in \mathcal{K}, \quad (4.7)$$

and one can use (3.16a) (with $N = 0, k = E$) to deduce that

$$H_0(E) = -i \int q^\alpha q^{-\alpha} \quad (4.8)$$

(summation implied). Next, observe that

$$\begin{aligned} \int \text{Tr}(E[A_x^0, \partial_{n, g} A_x^0]) &= i \int (q_{n, g}^\alpha q^{-\alpha} - q^\alpha q_{n, g}^{-\alpha}) \\ &= 2i \int q_{n, g}^\alpha q^{-\alpha} = -2i \int q_{n, g}^{-\alpha} q^\alpha, \end{aligned} \quad (4.9)$$

if $g \in \mathcal{K}$ (using (4.7), (4.8)). Differentiation of (4.9) with respect to $q^{\pm \alpha}$ gives Hamilton's equations (4.3), so that one can use (3.16) to write

$$H_N(g) = -ia \int \text{Tr}(A_x^0 \omega g \omega^{-1})_{N+1}, \quad (4.10a)$$

$$H_{-N}(g) = ia \int \text{Tr}(A_x^0 \psi (\Omega g \Omega^{-1})_{N-1} \psi^{-1}), \quad (4.10b)$$

where $a = 1/2$ if $g \in \mathcal{K}$, and $a = 1$ if $g \in \mathcal{M}$.

5. Linearization

For step operators $e_{\pm \alpha} \in \mathcal{M}$, define the formal power series

$$\Gamma_{\pm \alpha}(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^{-(n+1)} H_n(e_{\pm \alpha}) = i \int \text{Tr}(A_x^0 (\psi \Omega e_{\pm \alpha} \Omega^{-1} \psi^{-1} - \omega e_{\pm \alpha} \omega^{-1})) \quad (5.1)$$

(by (4.10)). Then (4.6) and (1.2) imply

$$\partial_{N, E} \Gamma_{\pm \alpha}(\lambda) = \{\Gamma_{\pm \alpha}(\lambda), H_N(E)\} = \pm i \lambda^N \Gamma_{\pm \alpha}(\lambda), \quad (5.2)$$

and so $\Gamma_{\pm \alpha}(\lambda)$ linearizes the equations of motion of the GNLS hierarchy.

Using the cyclic property of the trace, (5.1) can be written as

$$\Gamma_{\pm \alpha}(\lambda) = i \int ((\Omega^{-1} \psi^{-1} A_x^0 \psi \Omega)^{\mp \alpha} - (\omega^{-1} A_x^0 \omega)^{\mp \alpha}), \quad (5.3)$$

where $(X)^{\mp \alpha}$ denotes the $e_{\mp \alpha}$ component of X . Now, the restriction to the compact or non-compact form corresponds to

$$(X)^{\alpha*} = \mp (X)^{-\alpha} \quad (5.4)$$

so that

$$\{\Gamma_\alpha(\lambda), \Gamma_\beta^*(\mu)\} = \mp \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \lambda^{-n-1} \mu^{-m-1} H_{n+m}([e_\alpha, e_{-\beta}]) \quad (5.5)$$

when $q^{-\alpha} = \mp q^{\alpha*}$. Also

$$\{\Gamma_\alpha(\lambda), \Gamma_\beta(\mu)\} = 0 \quad (5.6)$$

(since (1.2) implies that $[e_\alpha, e_\beta] = 0$). Equation (5.5) shows that the transformation

$$q^\alpha \rightarrow \Gamma_\alpha \quad (5.7)$$

is not canonical.

6. Discussion

Recalling Eq. (3.14), one notes the following special cases of (3.29):

$$[\partial_x, \partial_{n,k}] = 0 \quad \forall n \in \mathbb{Z}, k \in \mathcal{L}, \quad (6.1a)$$

$$[\partial_x, \partial_{n,e_{\pm\alpha}}] = \mp i \partial_{1+n,e_{\pm\alpha}} \quad \forall n \in \mathbb{Z}, e_{\pm\alpha} \in \mathcal{M}. \quad (6.1b)$$

The parameters (n, k) could be regarded as “time” variables, but $(n, e_{\pm\alpha})$ cannot. Transformations which do not commute with translations are regarded, in the context of gauge theories, as “internal symmetries” [4]. The mutually commuting class $\{\partial_{n,e_\alpha}\}$ can be thought of as the basis of an “internal space.” It is the generator of translations in this space (Γ_α) which linearizes the GNLS system. It should be noted, however, that what one is really considering is the phase space of the system. The construction seems, in fact, to be a generalization of the conventional approach to the $SU(2)$ non-linear Schrödinger equation [5], where one considers the so-called “monodromy matrix” whose diagonal elements give rise to conserved quantities, while the off-diagonal elements lead to the linearization of the system.

The use of the gauge transformation $\tilde{\omega}$ is similar to the method used by Olive and Turok for deriving the conserved quantities of the Toda equation [6]. In that case, a λ -independent local gauge transformation was composed with a local transformation of the form (1.25) so that the transformed gauge potential was a series belonging to $\mathcal{L}$. In the present case, as was mentioned earlier, the λ -independent element ψ is associated with the generalized Heisenberg ferromagnet (GHF) [2]. Consider the transformation

$$A_x \rightarrow \tilde{A}_x = \psi^{-1} A_x \psi + \psi^{-1} \psi_x = \lambda \psi^{-1} E \psi, \quad (6.2a)$$

$$A_2(E) \rightarrow \tilde{A}_t = \psi^{-1} A_2(E) \psi + \psi^{-1} \psi_{2,E} = \lambda^2 \psi^{-1} E \psi + \lambda \psi^{-1} A_x^0 \psi \quad (6.2b)$$

(by (3.10)) and define

$$S = \psi^{-1} E \psi. \quad (6.3)$$

Then

$$\partial_x S = \psi^{-1} [E, \psi_x \psi^{-1}] \psi = \psi^{-1} [A_x^0, E] \psi, \quad (6.4)$$

and so

$$[S, S_x] = \psi^{-1} [E [A_x^0, E]] \psi = \psi^{-1} A_x^0 \psi, \quad (6.5)$$

i.e. the transformed gauge potentials are

$$\tilde{A}_x = \lambda S, \quad (6.6a)$$

$$\tilde{A}_t = \lambda^2 S + \lambda [S, S_x], \quad (6.6b)$$

and the zero curvature condition becomes the GHF equation

$$\partial_t S = [S, S_{xx}]. \quad (6.7)$$

The conserved quantities which have been constructed for the GNLS system are non-local, because of the non-locality of the gauge transformations used to construct them. Non-local conserved quantities were constructed for the non-linear σ -model in [7], and it was shown in [8] that these are associated with infinitesimal transformations which form a centre-free Kac-Moody algebra. However, the charges themselves do not form an algebra [9], and the Kac-Moody symmetry is interpreted as a property of the solution space, rather than of the phase space. The infinitesimal symmetries of the $SU(2)$ non-linear Schrödinger equation were investigated in [10]. In that construction, only the “positive” subalgebra was realized non-trivially.

The linearization of the GNLS system using step operators of a Kac-Moody algebra (Eq. (5.2)) seems to be related to the work of the Kyoto group [11], who use vertex operators to construct soliton solutions for a large class of equations. It would be interesting to establish the connection of these ideas with the approach of Adler and van Moerbeke [12]. Other topics worth pursuing include the investigation of the central term in (4.6) (e.g. the conditions under which it vanishes), and the quantization of the system. For the $SU(2)$ case, the quantization of the action-angle variables (i.e. the canonical linearizing variables) gives the “Bethe ansatz” creation operators [13]. In the general case, quantization should lead to vertex operators of some sort.

The methods which have been presented here can be generalized to cover a wide range of integrable systems. This will be discussed in a subsequent paper [14].

Acknowledgements. This work was carried out under the supervision of Professor D. I. Olive, with financial support from the S.E.R.C.

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Communicated by K. Osterwalder

Received September 25, 1986

