INTEGRABLE DYNAMICAL SYSTEMS ASSOCIATED WITH KAC-MOODY ALGEBRAS

by

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ABSTRACT

In Chapter 2 it is shown how to construct infinitely many conserved quantities for the classical non-linear Schrödinger equation associated with an arbitrary Hermitian symmetric space G/K. These quantities are non-local in general, but include a series of local quantities as a special case. Their Poisson bracket algebra is studied, and is found to be a realization of the "half" Kac-Moody algebra $\underset{\sim}{k} \otimes \mathbb{C}[\lambda]$, consisting of polynomials in positive powers of a complex parameter λ which have coefficients in $\underset{\sim}{k}$ (the Lie algebra of K).

In Chapter 3 the construction is extended to provide a realization of the Kac-Moody algebra $\underset{\sim}{k} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. One can then define auxiliary quantities to obtain the full algebra $\underset{\sim}{g} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. This leads to the formal linearization of the system.

In Chapter 4 the procedure is generalized so as to enable one to construct realizations of centre-free Kac-Moody algebras as hierarchies of 1+1 dimensional classical dynamical systems. The equations of motion (which are, in general, non-local) have Hamiltonians which form realizations of the same algebra. The Cartan subalgebra provides infinitely many conserved quantities in involution, while a sub-class of the step operators (which may be interpreted as generators of translations in "internal dimensions") enable the systems to be linearized. The systems can be regarded as having a "gauge symmetry" which includes momentum.

PREFACE

The work in this thesis was carried out in the Theoretical Physics group at the Department of Physics, Imperial College, London, between October 1983 and October 1987 under the supervision of Professor D.I. Olive. Financial support was provided by the SERC from October 1983 until September 1986. Unless otherwise stated, the work is original, and it has not been submitted before for a degree of this or any other university.

The thesis consists of three papers. The first two appeared in Communications in Mathematical Physics 108, 631-646 (1987); 111, 167-179 (1987), and the third has been submitted for publication (Imperial Preprint TP/87-88/2).

I should like to express my gratitude to Professor Olive for introducing me to so fruitful an area of research, and to thank him for his help.

"of the said to made it has a good road"

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CHAPTER 1: INTRODUCTORY REMARKS

A classical (conservative) Hamiltonian system with n degrees of freedom and Poisson bracket $\{\ ,\ \}$ is called completely integrable if there exist n mutually involutive conserved quantities I_j; i.e.

$$\partial_t I_j = \{H, I_j\} = 0 \qquad 1 \leq j \leq n \tag{1}$$

$$\{I_j, I_k\} = 0 \qquad 1 \leq j,k \leq n \tag{2}$$

Under these conditions one can (in principle) perform a transformation to new canonical variables I_j, ϕ_j ("action-angle" variables). The Hamiltonian will be expressible in terms of the I_j variables alone, and so the Hamiltonian equations for ϕ_j can be integrated to give

$$\phi_j(t) = \phi(0) + (\partial H/\partial I_j)t \tag{3}$$

and the system can be solved by transforming back to the original dynamical variables. Systems of this type have the maximum possible degree of constraint, and can be thought of as the opposite of totally disordered (ergodic) systems.

The concept first arose in the nineteenth century, particularly in the work of Jacobi, who investigated the case of geodesic motion on an ellipsoid, and other systems [1]. The integration of the equations (by "quadratures") gave rise to the theory of elliptic functions. It should be noted that even if the quantities I_j are not in involution, it may still be possible to obtain action-angle variables and solve the system; for example, the six conserved quantities in the

Kepler problem have a Poisson bracket algebra isomorphic to SO(4) [2]. The action-angle variables for this system are of importance in celestial mechanics, and also played a role in the development of atomic theory. Some authors only require the existence of action-angle variables for a system to be called completely integrable.

Until quite recently, few cases were known of non-linear systems which are integrable, but during the last twenty years it has become apparent that a wide range of physically interesting examples do exist. Many of these are (one dimensional) field theories; i.e. there are infinitely many conserved quantities.

The modern study of integrable systems could be said to have been initiated (fortuitously) in a computer experiment of Fermi, Pasta, and Ulam [3] on the behaviour of one dimensional lattices (a line of masses linked by "springs") with linear nearest-neighbour interactions which are subject to a nonlinear perturbation. For purely linear interactions, the energy in each normal mode will remain constant in time. One would expect the perturbation (which introduces non-linear coupling between the modes) to give rise to ergodic motion (i.e. the energy eventually becomes evenly distributed between the modes). Surprisingly the behaviour of the lattice was in fact found to be periodic! If the non-linear perturbation is quadratic, then one can, with certain approximations, transform the equation of motion for a lattice particle in the continuum limit to the Korteweg-de Vries (KdV) equation:

$$u_t + \alpha u u_x + u_{xxx} = 0 \qquad (4)$$

(see e.g. [4]). Eq. (4) can be regarded as the generic example of a system which is weakly dispersive and weakly non-linear. It was used by Korteweg and de Vries [5] as a model for water waves in a shallow channel. A solution of (4) is

$$u = \operatorname{sech}^2(\lambda x + \lambda^3 t) \tag{5}$$

This is a localised "hump" which maintains its shape as it propagates. (The first observation in nature of a wave of this form was made by Russell in 1834 on the Union Canal near Edinburgh). In the lattice experiment, momentum was being transferred along the chain in localized "packets". This was the first indication that the KdV equation is integrable.

Further numerical studies were carried out by Zabusky and Kruskal [6], who found more "solitary wave" solutions, and discovered that these had the remarkable property that when they passed through each other they only underwent a change of phase. They named these waves "solitons", to reflect their particle-like behaviour.

The complete analytical solution of the KdV equation was carried out by Gardner et al in 1967 [7]. They showed that if u is a solution then the Schrödinger equation

$$\psi_{xx} - u\psi = \lambda\psi \tag{6}$$

has time independent eigenvalues. Then the problem was reduced to that of reconstructing a "potential" u for the Schrödinger equation given a set of scattering data (boundary conditions). This problem had previously been solved by Gel'fand and

Levitan [8]. Because of the connection with scattering theory, the method of solution is known as "inverse scattering". One consequence of the method was that it provided a construction for infinitely many conserved quantities, which indicated that the KdV equation can be regarded as an integrable field theory.

It was shown by Lax [9] that the feature underlying the inverse scattering method is the representation of the KdV equation in the form

$$\partial_t L = [L,A] \tag{7}$$

for operators L,A which depend on $(x,\partial_x,\partial_{xx},\ldots)$. By substituting other "Lax pairs" L,A into (7), one could obtain (and solve) other systems. The solution of the system is possible because the "scattering data" evolves linearly. This is reminiscent of the transformation to action-angle variables mentioned earlier. It has been shown by Zakharov and Faddeev [10] that they are in fact related.

Another important reformulation of the method was introduced by Zakharov and Shabat (ZS) [11], who considered the non-linear Schrödinger equation

$$iq_t = q_{xx} - 2|q|^2 q^* \tag{8}$$

which arises (e.g. in optics) when there is weak non-linearity and strong dispersion. They observed that (8) can be written as

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0 \tag{9}$$

where

$$A_x = -i \begin{pmatrix} \lambda/2 & q^* \\ -q & -\lambda/2 \end{pmatrix} \tag{10}$$

$$A_t = -i \begin{pmatrix} -\lambda^2/2 - q^* q & -\lambda q^* + i q_x^* \\ \lambda q + i q_x & \lambda^2/2 + q^* q \end{pmatrix} \tag{11}$$

(notice that the parameter λ does not appear in the equation of motion). The representation (9) was used by Ablowitz et al (AKNS) [12] to solve a wide class of systems. It is the ZS/AKNS representation which will be used (and further generalized) in this thesis, rather than the operator formalism of Lax.

Eq. (9) can be regarded as a "zero-curvature condition" for "gauge potentials" (A_x, A_t), and is invariant under a "gauge transformation"; i.e. if ω is any 2×2 unimodular matrix and

$$a_\mu = \omega^{-1} A_\mu \omega + \omega^{-1}(\partial_\mu \omega) \tag{12}$$

then

$$\partial_x a_t - \partial_t a_x + [a_x, a_t] = 0 \tag{13}$$

and (13) is a new representation for the equation of motion. This fact enables one to construct the conserved quantities in a simple way, by choosing ω to be an infinite polynomial in powers of λ (with matrix coefficients) such that a_x and a_t are diagonal. Then

$$\partial_x a_t - \partial_t a_x = 0 \tag{14}$$

and the coefficients of powers of λ in a_x are conserved currents, i.e. if a_t is expressed in terms of fields at a point (and their derivatives) then

$$\partial_t \int a_x = 0 \tag{15}$$

(using integration by parts). More generally, if (A_x, A_t) are polynomials in powers of λ with coefficients in a Lie algebra $\underset{\sim}{g}$, then one wishes to find a group valued element ω such that a_x, a_t belong to a Cartan subalgebra of $\underset{\sim}{g}$. This has been carried out for the non-linear σ model in [13], and the Toda system in [14].

The "gauge symmetry" of these systems suggests a link with high energy physics; indeed, the Toda equation arises in the theory of magnetic monopoles [15], and the self dual Yang Mills gauge theory has been shown to possess an infinite class of conserved quantities [16]. Of greatest interest, however, is the possibility of a connection with string theory. It has recently been observed that the work of several Japanese authors [17] on the completely integrable Kadomtsev-Petviashvili equation is of relevance in string theory, and this may lead one to wonder whether integrability may be a fundamental natural principle.

The starting point for the work presented here was an attempt to construct (in the manner of [14]) the conserved quantities associated with a matrix generalization of (8) due to Fordy and Kulish [18]:

$$iq_t^\alpha = q_{xx}^\alpha \pm q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta} \tag{16}$$

(summation implied over repeated indices, and $e_\alpha R^\alpha_{\beta\gamma-\delta} = [e_\beta[e_\gamma, e_{-\delta}]]$), which is here called the "generalized non-linear Schrödinger" (GNLS) equation. The A_x potential is of the form

$$A_x = \lambda E + A_x^0 \tag{17}$$

with coefficients in a "symmetric Lie algebra"

$$\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m} \tag{18}$$

$$[\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \qquad [\underset{\sim}{k},\underset{\sim}{m}] \subset \underset{\sim}{m} \qquad [\underset{\sim}{m},\underset{\sim}{m}] \subset \underset{\sim}{k} \tag{19}$$

A_x^0 is a matrix field which lies in $\underset{\sim}{m}$, and E is a special element such that

$$\underset{\sim}{k} = \{g \in \underset{\sim}{g}: [E,g] = 0\} \tag{20}$$

and $(\text{ad}E)^2 = -I$ on $\underset{\sim}{m}$. A_x given by (17) is a generalization of the potential (10), and one can look for the corresponding generalization of (11) by solving the zero curvature condition, i.e. (following [18]), choose

$$A_t = \lambda^2 A_t^{(2)} + \lambda A_t^{(1)} + A_t^{(0)} \tag{21}$$

and substitute A_x, A_t into

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0 \tag{22}$$

Now equate coefficients of powers of λ. For λ^3 one obtains

$$[E, A_t^{(2)}] = 0 \tag{23}$$

i.e. $A_t^{(2)} \in \underset{\sim}{k}$, and λ^2 gives

$$\partial_x A_t^{(2)} + [E, A_t^{(1)}] + [A_x^0, A_t^{(2)}] = 0 \tag{24}$$

Using (19) to split this into $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components this becomes

$$\partial_x (A_t^{(2)}) = 0 \tag{25}$$

$$[E, A_t^{(1)}] + [A_x^0, A_t^{(2)}] = 0 \tag{26}$$

i.e. $A_t^{(2)}$ is a constant element of $\underset{\sim}{k}$, and one can obtain $(A_t^{(1)})^{\underset{\sim}{m}}$ from (26). Then, from the coefficient of λ^1,

$$\partial_x (A_t^{(1)})^{\underset{\sim}{k}} + [A_x^0, (A_t^{(1)})^{\underset{\sim}{m}}] = 0 \tag{27}$$

$$\partial_x (A_t^{(1)})^{\underset{\sim}{m}} + [E, A_t^{(0)}] + [A_x^0, (A_t^{(1)})^{\underset{\sim}{k}}] = 0 \tag{28}$$

One obtains $(A_t^{(1)})^{\underset{\sim}{k}}$ from (27), and $(A_t^{(0)})^{\underset{\sim}{m}}$ from (28). The coefficient of λ^0 gives

$$\partial_x (A_t^{(0)})^{\underset{\sim}{k}} + [A_x^0, (A_t^{(0)})^{\underset{\sim}{m}}] = 0 \tag{29}$$

$$\partial_t A_x^0 = \partial_x (A_t^{(0)})^{\underset{\sim}{m}} + [A_x^0, (A_t^{(0)})^{\underset{\sim}{k}}] \tag{30}$$

$(A_t^{(0)})^{\underset{\sim}{k}}$ is found from (29), and the corresponding equation of motion is given by (30).

If one chooses $A_t^{(2)} = E$ then the solution can be reduced to

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]] \qquad (31)$$

Writing $A_x^0 = -q^\alpha e_\alpha \pm q^{\alpha *} e_{-\alpha}$, the corresponding equation of motion is (16). For a general choice of $A_t^{(2)}$, however, A_t is non-local (the general solution of (27) is $(A_t^{(1)})^{\underset{\sim}{k}} = -\int_{-\infty}^{x} dx[A_x^0, (A_t^{(1)})^{\underset{\sim}{m}}])$, and the equation of motion is a mixed integro-differential equation.

Instead of choosing A_t to be a polynomial of order two, one can solve (22) to find A_t as a polynomial to any order N in positive powers of λ. One finds in the same way that the coefficients of degree less than N-1 are, in general, non-local. The equations of motion form a "hierarchy" associated with A_x given by (17), each labelled by (N,k) (the order of A_t, and the choice of $A_t^{(2)}$). The GNLS equation is labelled (2,E). Now, an observed feature of 2×2 ZS-type systems is that the Hamiltonians of such hierarchies are conserved quantities for any equation of motion in the hierarchy, and this suggests that the GNLS equation may have non-local conserved quantities (which do not arise in the 2×2 case, since $\underset{\sim}{k}$ is one dimensional and one must choose $A_t^{(2)}$ proportional to E). One can construct a non-local gauge transformation which takes A_x, A_t into the Cartan subalgebra, but then the equation (14), being non-local, may no longer be interpreted as a continuity equation. So the problem which will first be considered, in Chapters 2 and 3, is the construction of Hamiltonians for the hierarchy of equations of motion associated with (17), and the derivation of their Poisson bracket algebra. This will motivate a general approach to integrable systems associated with zero curvature representations, which is presented in Chapter 4.

REFERENCES

[1] Jacobi, C.G.J: Vorlesungen uber Dynamik. Berlin (1884). (Reprinted in Mathematische Werke vol. 8, Chelsea Publishing Company, New York 1969)

[2] See e.g. Goldstein, H.: Classical Mechanics, 2nd ed. Addison-Wesley (1980).

[3] Fermi, E., Pasta, J.R., Ulam, S.M.: Studies of nonlinear problems. Technical Report LA-1940, Los Alamos Sci. Lab. (1955)

[4] Dodd, R.K., Eilbeck, J.C., Gibbon, J.D., Morris, H.C.: Solitons and nonlinear wave equations. Academic Press (1982)

[5] Korteweg, D.J., de Vries, G.: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Phil. Mag. 39, 422-443 (1895)

[6] Zabusky, N.J., Kruskal, M.D.: Interaction of solitons in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15, 240-243 (1965)

[7] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. 19, 1095-1097 (1967)

[8] Gel'fand, I.M., Levitan, B.M.: On the determination of a differential equation from its spectral function. Amer. Math. Soc. Transl. 1, 253-304 (1955)

[9] Lax, P.D.: Integrals of nonlinear equations of evolution and solitary waves. Commun. Pure and Appl. Math. 21, 467-490 (1968)

[10] Zakharov, V.E., Faddeev, L.D.: Korteweg-de Vries equation: a completely integrable Hamiltonian system. Funct. Anal. Appl. 5, 280-287 (1971)

[11] Zakharov, V.E., Shabat, A.B.: Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media. Sov. Phys. JETP 34, 62-69 (1972)

[12] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Non-linear equations of physical significance. Phys. Rev. Lett. 31, 125-127 (1973)

[13] Eichenherr, H., Forger, M.: Higher local conservation laws for nonlinear sigma models on symmetric spaces. Commun. Math. Phys. 82, 227-255 (1981)

[14] Olive, D., Turok, N.: Local conserved densities and zero curvature conditions for Toda lattice field theories. Nucl. Phys. B257[FS14], 277 (1985)

[15] Leznov, A.M., Saveliev, M.V.: Representation theory and integration of nonlinear spherically symmetric equations to gauge theories. Commun. Math. Phys. 74, 111-118 (1980).

[16] For a review, see Dolan, L.: Kac-Moody algebras and exact solvability in hadronic physics. Phys. Rep. 109 No. 1, 1-94 (1984).

[17] For reviews, see Jimbo, M., Miwa, T.: Solitons and infinite dimensional Lie algebras. Publ. RIMS, Kyoto Univ. 19, 943-1001 (1983); and Segal, G., Wilson, G.: Loop groups and equations of KdV type. Publ. I.H.E.S. 61, 5-65 (1985)

[18] Fordy, A.P., Kulish, P.P.: Nonlinear Schrödinger equations and simple Lie algebras. Commun. Math. Phys. 89, 427-443 (1983)

CHAPTER 2: LOCAL AND NON-LOCAL CONSERVED QUANTITIES FOR GENERALIZED NON-LINEAR SCHRÖDINGER EQUATIONS.

1. INTRODUCTION.

Fordy and Kulish [1] have considered a class of non-linear partial differential equations, each associated with an Hermitian symmetric space G/K, which are of the form

$$iq_t^\alpha = q_{xx}^\alpha - q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta} \tag{1.1}$$

where summation is implied over repeated indices. $q^\alpha(x,t)$ are fields in one space dimension whose label α denotes a root of $\underset{\sim}{g}$ (the Lie algebra of G) such that the step operator e_α does not lie in $\underset{\sim}{k}$ (the Lie algebra of K). R is the "curvature tensor" defined by

$$e_\alpha R^\alpha_{\beta\gamma-\delta} = [e_\beta[e_\gamma, e_{-\delta}]] \tag{1.2}$$

A special case of (1.1), corresponding to G=SU(2), is the non-linear Schrödinger (NLS) equation

$$iq_t = q_{xx} + 2|q|^2 q \tag{1.3}$$

Equation (1.1) will be referred to as the Generalized non-linear Schrödinger (GNLS) equation associated with G/K. The NLS equation is known to have infinitely many conserved quantities which are local (in the sense that the currents are expressed only in terms of the fields $q(x,t)$, $q^*(x,t)$ and their derivatives at a point), and are in involution (i.e. their Poisson bracket algebra is abelian). The aim of this paper is to construct the algebra of conserved quantities for the GNLS equation.

The existence of such quantities is related to the fact that the equation of motion can be expressed as a "zero curvature condition"

$$F_{xt} \equiv [\partial_x + A_x, \partial_t + A_t] = 0 \tag{1.4}$$

where A_x, A_t are Lie algebra valued polynomials in a parameter $\lambda \in \mathbb{C}$ (the "spectral parameter") which does not appear in the equation of motion. Equation (1.4) is the consistency condition for the coupled pair of linear equations

$$\Phi_x + A_x\Phi = 0 \tag{1.5a}$$

$$\Phi_t + A_t\Phi = 0 \tag{1.5b}$$

For the NLS equation, A_x and A_t are 2x2 matrices, and it is fairly easy to construct the group element Φ (the "monodromy matrix"). The logarithm of its diagonal elements can be expanded in powers of λ to give conserved quantities [2].

It is shown in [1] that the GNLS equation is associated with the pair

$$A_x = \lambda E + A_x^0 \tag{1.6a}$$

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]] \tag{1.6b}$$

where E is a special constant element which commutes with any element of $\underset{\sim}{k}$, satisfying

$$[E,e_\alpha] = -ie_\alpha \qquad \text{for all } \alpha \tag{1.7}$$

and

$$A_x^0 \equiv -q^\alpha e_\alpha + q^{\alpha *} e_{-\alpha} \tag{1.8}$$

For algebras of rank greater than one, the monodromy matrix (which is a path ordered exponential) becomes difficult to work with. It is then more convenient to use the algebraic properties of the zero curvature condition (1.4), in particular its invariance under a gauge transformation

$$A_x \rightarrow a_x = \omega^{-1} A_x \omega + \omega^{-1} \omega_x \tag{1.9a}$$

$$A_t \rightarrow a_t = \omega^{-1} A_t \omega + \omega^{-1} \omega_t \tag{1.9b}$$

where $\omega \in G$. The new pair a_x, a_t are associated with the same equation of motion as the pair A_x, A_t. Olive and Turok [3] have used this invariance to study the Toda equation. In that case it is possible to construct ω so that $a_x, a_t \in \underset{\sim}{h}$, the Cartan subalgebra. ω takes the form

$$\omega = \exp\Sigma_{n=1}^{\infty} \lambda^{-n} \omega_n \tag{1.10}$$

and is local. Then a_x and a_t are descending power series, and the zero curvature condition becomes

$$\partial_x a_t - \partial_t a_x = 0 \tag{1.11}$$

which implies that the coefficients of arbitrary powers of λ are conserved currents.

In attempting to apply this method to the GNLS equation, one encounters the problem that the gauge transformation which takes A_x and A_t into the Cartan subalgebra is now non-local, so that equation (1.11) can no longer be interpreted as a conservation law. In order to discuss non-local conserved quantities, it is necessary to investigate the Poisson bracket algebra.

Consider first the Hamiltonian of the GNLS equation. Using (1.7) and (1.8) one finds

$$\mathrm{Tr}([E,A_x^0]\partial_t A_x^0) = iq^\alpha q_t^{\alpha *} - iq_t^\alpha q^{\alpha *} \tag{1.12}$$

If q^α, $q^{\alpha *}$ are regarded as canonical variables, then differentiation of both sides of (1.12) with respect to q^α, $q^{\alpha *}$ gives Hamilton's equations

$$q_t^\alpha = \partial H/\partial q^{\alpha *} \tag{1.13a}$$

$$q_t^{\alpha *} = -\partial H/\partial q^\alpha \tag{1.13b}$$

where

$$H \propto i\int \mathrm{Tr}([E,A_x^0]\partial_t A_x^0) \tag{1.14}$$

(the proportionality sign is used because there is actually a constraint which must be taken into account).

Now, the equation of motion can be read off from the zero curvature condition (1.4) as the coefficient of λ^0:

$$\partial_t A_x^0 = \partial_x A_t^0 + [A_x^0, A_t^0] \tag{1.15}$$

where A_t^0 is the coefficient of λ^0 in (1.6b). In this way one obtains an explicit expression for H in terms of the fields q^α, $q^{\alpha *}$ and their derivatives.

It was shown in [1] that instead of considering A_t given by (1.6b), one can look for

$$A_t = \Sigma_{n=0}^{N} \lambda^n A_t^n \tag{1.16}$$

by substituting into the zero curvature condition (1.4) and equating coefficients of λ^n to zero. The coefficient of the highest power of λ, i.e. A_t^N, is left undetermined, but must be a constant element of $\underset{\sim}{k}$. When $A_t^N = E$, the resulting expression for $\bar{A}_t$ is local, but for a general element $A_t^N = k \in \underset{\sim}{k}$, one finds that A_t is non-local. $A_N(k)$ will denote A_t having the leading term $\lambda^N k$.

Each possible choice of $A_N(k)$ will give rise to a different equation of motion, given by the coefficient of λ^0 in the zero curvature condition:

$$\partial_{N,k} A_x^0 = \partial_x A_N^0(k) + [A_x^0, A_N^0(k)] \tag{1.17}$$

The collection of operators $\partial_{N,k}$ will be regarded as independent evolution operators defining infinitely many "times". When N=2 and k=E, $A_N(k)$ is given by (1.6b), and so $\partial_{2,E}$ is the GNLS evolution operator. For a fixed value of k one has a hierarchy of equations of motion labelled by $N\geq 0$. When k=E this will be referred to as the "GNLS hierarchy".

For each equation of motion (1.17), one can obtain its Hamiltonian in the form (1.14). The Hamiltonian for the equation arising from the pair A_x, $A_N(k)$ will be denoted by $H_N(k)$. It will be seen that $H_N(k)$ is non-local in general, but the Hamiltonians $H_N(E)$ of the GNLS hierarchy are local. Furthermore, the entire collection of $H_N(k)$ will turn out to be conserved quantities for the GNLS equation. To show this it is necessary to construct the Poisson bracket algebra of the Hamiltonians, and this is done by considering the commutation relations of the evolution operators $\partial_{N,k}$. One first has to find closed expressions for $\partial_{N,k}$ and $A_N(k)$ (the method used in [1] of solving the zero curvature condition gives the coefficients of $A_N(k)$ recursively). This is where the gauge invariance property proves useful. It turns out that a non-local transformation of the form (1.10) can be constructed so that

$$A_x \rightarrow a_x = \lambda E \tag{1.18}$$

The zero curvature condition will then be satisfied by

$$a_t = \lambda^N k \equiv a_N(k) \tag{1.19}$$

where $N\geq 0$ and $k \in \underset{\sim}{k}$ is constant. Now the gauge transformation

(1.9b) is inverted to obtain

$$A_t = \omega a_N(k)\omega^{-1} - \omega_{N,k}\omega^{-1}$$

$$= \lambda^N \omega k \omega^{-1} - \omega_{N,k}\omega^{-1} \qquad (1.20)$$

If A_t is chosen to have only positive powers of λ then one can equate coefficients to obtain

$$A_t = \Sigma_{n=0}^{N} \lambda^{N-n} (\omega k \omega^{-1})_n = A_N(k) \qquad (1.21)$$

(where $\omega k \omega^{-1} = \Sigma_{n=0}^{\infty} \lambda^{-n} (\omega k \omega^{-1})_n$). Also, one finds that the coefficient of λ^{-1} in (1.20) gives

$$\partial_{N,k} A_x^0 = -[E, (\omega k \omega^{-1})_{N+1}] \qquad (1.22)$$

which is the equation of motion associated with $A_N(k)$ in closed form. This will be used to derive the main result of this paper;

$$[\partial_{N,k} , \partial_{M,j}] A_x^0 = \partial_{N+M,[k,j]} A_x^0 \qquad (1.23)$$

for all $N, M \geq 0$, $k, j \in \underset{\sim}{k}$. In other words, the evolution operators form "half" of a Kac-Moody algebra (since N and M take only positive values). Equation (1.23) will be used, together with the Jacobi identity, to establish the final result

$$\{H_N(k), H_M(j)\} = H_{N+M}([k,j]) \qquad (1.24)$$

which states that the Hamiltonians have the same "Kac-Moody" algebraic structure under the Poisson bracket. In particular, one has

$$\{H_N(k), H_2(E)\} = 0 \tag{1.25}$$

This means that the entire collection of Hamiltonians are conserved quantities for the GNLS equation.

In Section 2 it will be shown how the gauge transformation ω is constructed in terms of the field variables q^{α}, $q^{\alpha*}$. The solution of the zero curvature condition to give $A_N(k)$ and $\partial_{N,k}$ in closed form will be discussed in Section 3. In Section 4 the Hamiltonians and their Poisson bracket algebra will be considered, and it will be shown in Section 5 that $H_N(E)$ is local for all N. Finally, in Section 6, the results obtained will be compared with the work of Olive and Turok, and possible generalizations to other systems will be discussed.

2. CONSTRUCTION OF ω.

Let G/K be an Hermitian symmetric space, where $\underset{\sim}{k}$ (the Lie algebra of K) is the centralizer of E and $\underset{\sim}{g}$ (the Lie algebra of G) decomposes as

$$\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m} \tag{2.1}$$

The step operators of the Cartan-Weyl basis of $\underset{\sim}{g}$ which lie in

$\underset{\sim}{k}$ are denoted by Latin letters (e_a), while those which lie in $\underset{\sim}{m}$ are denoted by Greek letters (e_α). The set of positive roots whose step operators lie in $\underset{\sim}{m}$ is called θ^+.

It is explained in the Appendix that E satisfies the property

$$[E,e_\alpha] = \kappa e_\alpha \tag{2.2}$$

where κ is a constant for all $\alpha \in \theta^+$. E will be chosen so that

$$\kappa = -i \tag{2.3}$$

Now, following [1], define

$$A_x = \lambda E + A_x^0 \tag{2.4}$$

where

$$A_x^0 = -q^\alpha e_\alpha + q^{\alpha *} e_{-\alpha} \in \underset{\sim}{m} \tag{2.5}$$

The main object of interest is the "zero curvature condition"

$$F_{xt} \equiv [\partial_x + A_x, \partial_t + A_t] = 0 \tag{2.6}$$

where A_t is a polynomial in λ with coefficients in $\underset{\sim}{g}$. The only restriction on A_t is that the resulting equation of motion for the fields $q^\alpha(x,t)$ implied by (2.6) must be independent of λ.

Equation (2.6) is invariant under a "gauge transformation"

$$A_\mu \to a_\mu = \omega^{-1}A_\mu\omega + \omega^{-1}\partial_\mu\omega \qquad \mu = (x,t) \tag{2.7}$$

where $\omega(\lambda;x,t) \in G$. In other words

$$[\partial_x+a_x,\partial_t+a_t] = 0 \tag{2.8}$$

Equation (2.8) is associated with the same equation of motion as (2.6). However, it may be possible to find a transformation such that a_μ is independent of the fields q^α, $q^{\alpha *}$. In that case, the equation of motion is implied by the transformation (2.7) with μ=t, which can be thought of as an equation of motion for ω. Such a transformation will, in fact, prove to be very useful in what follows. Notice that A_x and A_t can be thought of as elements of the "loop algebra" $\underset{\sim}{g} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$ (where $\mathbb{C}[\lambda,\lambda^{-1}]$ is the algebra of Laurent polynomials in the complex variable λ). It is therefore natural to consider ω as an element of the "loop group". It will be chosen to have the form

$$\omega = \exp\Sigma_{n=1}^{\infty}\lambda^{-n}\omega_n \qquad (\omega_n \in \underset{\sim}{g}) \tag{2.9}$$

This is the type of gauge transformation used in [3] in connection with the Toda equation. By expanding (2.9) as a power series in λ one obtains the identities:

$$(\omega^{-1}A\omega)_n = \Sigma_{r=1}^{n}(-1)^r(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\omega_{k_2}[\ldots[\omega_{k_r},A]\ldots]]] \tag{2.10a}$$

$$(\omega A\omega^{-1})_n = \Sigma_{r=1}^{n}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\omega_{k_2}[\ldots[\omega_{k_r},A]\ldots]]] \tag{2.10b}$$

$$(\omega^{-1}\omega_\mu)_n =$$

$$\Sigma_{r=1}^{n}(-1)^{r+1}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\ldots[\omega_{k_{r-1}},\partial_\mu\omega_{k_r}]\ldots]] \tag{2.11a}$$

$$(\omega_\mu\omega^{-1})_n =$$

$$\Sigma_{r=1}^{n}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\omega_{k_2}[\ldots[\omega_{k_{r-1}},\partial_\mu\omega_{k_r}]\ldots]]] \tag{2.11b}$$

These can then be used to write a_x (2.7) as a power series:

$$a_x = \lambda E + \Sigma_{n=0}^{\infty}\lambda^{-n}a_x^n$$

$$= \lambda E + \{A_x^0 - [\omega_1,E]\}$$

$$+ \lambda^{-1}\{ -[\omega_2,E] + 1/2[\omega_1[\omega_1,E]] - [\omega_1,A_x^0] + \partial_x\omega_1\}$$

$$+ \lambda^{-2}\{ -[\omega_3,E] + 1/2[\omega_1[\omega_2,E]] + 1/2[\omega_2[\omega_1,E]]$$

$$- 1/6[\omega_1[\omega_1[\omega_1,E]]] - [\omega_2,A_x^0] + 1/2[\omega_1[\omega_1,A_x^0]]$$

$$+ \partial_x\omega_2 - 1/2[\omega_1,\partial_x\omega_1] \} + \ldots \tag{2.12}$$

It will now be shown that it is possible to construct ω so that

$$a_x = \lambda E \tag{2.13}$$

One can see from (2.12) that $a_x^0 = 0$ if

$$[\omega_1, E] = A_x^0 \tag{2.14}$$

Using (A.8) and (2.3), this implies

$$\omega_1^{\underset{\sim}{m}} = [E, A_x^0]$$

$$= iq^\alpha e_\alpha + iq^{\alpha *} e_\alpha \tag{2.15}$$

where $\omega_1^{\underset{\sim}{m}}$ denotes the component of ω_1 in $\underset{\sim}{m}$. Now consider a_x^1. The commutation relations (A.14) can be used to equate the $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components to zero:

$$(a_x^1)^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_1, E]] - [\omega_1^{\underset{\sim}{m}}, A_x^0] + \partial_x \omega_1^{\underset{\sim}{k}} = 0$$

$$\text{i.e.} \quad \partial_x \omega_1^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}, A_x^0] \tag{2.16}$$

using (2.14), and

$$(a_x^1)^{\underset{\sim}{m}} = -[\omega_2, E] + 1/2[\omega_1^{\underset{\sim}{k}}[\omega_1, E]] - [\omega_1^{\underset{\sim}{k}}, A_x^0] + \partial_x \omega_1^{\underset{\sim}{m}} = 0$$

i.e.

$$[\omega_2, E] = \partial_x \omega_1^{\underset{\sim}{m}} - 1/2[\omega_1^{\underset{\sim}{k}}, A_x^0] \tag{2.17}$$

using (2.14) again. Notice that (2.16) determines $\omega_1^{\underset{\sim}{k}}$ non-locally:

$$\omega_1^{\underset{\sim}{k}} = 1/2\partial^{-1}[A_x^0[A_x^0, E]] \tag{2.18}$$

while (2.17) gives

$$\omega_2^{\underset{\sim}{m}} = -\partial_x A_x^0 + 1/2[E[A_x^0, \partial^{-1}[A_x^0[A_x^0, E]]]] \tag{2.19}$$

For a general term a_x^n (n>1) one has

$$a_x^n = -[\omega_{n+1}, E] + 1/2[\omega_1[\omega_n, E]] + 1/2[\omega_n[\omega_1, E]] - [\omega_n, A_x^0]$$
$$+ \partial_x \omega_n + (\text{terms involving } \omega_{j<n}) \tag{2.20}$$

This can be split up into $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components and equated to zero to obtain

$$\partial_x \omega_n^{\underset{\sim}{k}} = -1/2[\omega_1^{\underset{\sim}{m}}[\omega_n^{\underset{\sim}{m}}, E]] - 1/2[\omega_n^{\underset{\sim}{m}}[\omega_1^{\underset{\sim}{m}}, E]] + [\omega_n^{\underset{\sim}{m}}, A_x^0]$$
$$+ (\text{terms involving } \omega_{j<n}) \tag{2.21}$$

$$[\omega_{n+1}, E] = 1/2[\omega_1^{\underset{\sim}{k}}[\omega_n^{\underset{\sim}{m}}, E]] + 1/2[\omega_n^{\underset{\sim}{k}}[\omega_1^{\underset{\sim}{m}}, E]] - [\omega_n^{\underset{\sim}{k}}, A_x^0]$$
$$+ \partial_x \omega_n^{\underset{\sim}{m}} + (\text{terms involving } \omega_{j<n}) \tag{2.22}$$

So for each n the requirement $a_x^n = 0$ determines $\omega_n^{\underset{\sim}{k}}$ and $\omega_{n+1}^{\underset{\sim}{m}}$.

In [3], only the condition $(a_x^n)\tilde{}^m=0$ is imposed, so that $\omega_n^{\tilde{k}}$ is left undetermined and can be chosen to be zero to all orders. This gauge transformation will be denoted w. The first few terms are obtained from (2.12) as follows:

$$[w_1,E] = A_x^0 \qquad \text{i.e.} \quad w_1 = [E,A_x^0] \tag{2.23}$$

$$[w_2,E] = \partial_x w_1 \qquad \text{i.e.} \quad w_2 = -\partial_x A_x^0 \tag{2.24}$$

$$[w_3,E] = -1/6[w_1[w_1[w_1,E]]] + 1/2[w_1[w_1,A_x^0]] + \partial_x w_2$$

$$\text{i.e.} \quad w_3 = 1/3[E[w_1[w_1,A_x^0]]] - \partial_{xx} w_1 \tag{2.25}$$

and so on. Notice that, unlike ω, w is local to all orders.

3. SOLUTION OF THE ZERO CURVATURE CONDITION.

One wishes to find A_t such that the zero curvature condition (2.6) is satisfied with A_x given by (2.4). As in [1], A_t will be assumed to be of the form

$$A_t = \Sigma_{n=0}^{N} \lambda^n A_t^n \tag{3.1}$$

Consider the gauge transformed potentials

$$a_x = \omega^{-1} A_x \omega + \omega^{-1} \omega_x = \lambda E \tag{3.2a}$$

$$a_t = \omega^{-1} A_t \omega + \omega^{-1} \omega_t \tag{3.2b}$$

where ω is the gauge transformation constructed in Section 2. One can see from the identities (2.10a), (2.11a) that a_t is a descending power series of the form

$$a_t = \lambda^N a_t^{-N} + \ldots + a_t^0 + \lambda^{-1} a_t^1 + \ldots \tag{3.3}$$

Now substitute (3.2a), (3.3) into the zero curvature condition

$$\partial_x a_t - \partial_t a_x + [a_x, a_t] = 0 \tag{3.4}$$

and equate powers of λ to zero:

$$\lambda^{N+1}: \quad [E, a_t^{-N}] = 0 \qquad \text{i.e.} \quad a_t^{-N} \in \underset{\sim}{k} \tag{3.5}$$

$$\lambda^N: \quad \partial_x a_t^{-N} + [E, a_t^{1-N}] = 0 \tag{3.6}$$

Split this into parts in $\underset{\sim}{k}$ and $\underset{\sim}{m}$ to obtain the result that a_t^{-N} is a constant and $a_t^{1-N} \in \underset{\sim}{k}$. Continuing in this way one finds that all of the coefficients of a_t are constant elements of $\underset{\sim}{k}$. One can choose

$$a_t = \lambda^N k \tag{3.7}$$

Now invert the transformation (3.2b):

$$A_t = \omega a_t \omega^{-1} - \omega_t \omega^{-1}$$

$$= \lambda^N \omega k \omega^{-1} - \omega_t \omega^{-1} \tag{3.8}$$

Since $\omega_t\omega^{-1}$ has only negative powers of λ, and A_t is chosen to have no negative powers, it follows that

$$A_t = \Sigma_{n=0}^{N}\lambda^{n}(\omega k\omega^{-1})_{N-n} \tag{3.9}$$

where $(\omega k\omega^{-1})_m$ denotes the coefficient of μ^{-m} in $\omega(\mu)k\omega^{-1}(\mu)$. By (2.10b), A_t given by (3.9) has $\lambda^N k$ as its highest order term. Since N and k are arbitrary, the notation $A_N(k)$ will be used for the object defined by (3.9).

Turning now to the negative powers of λ in (3.8), one can equate coefficients to obtain

$$(\omega_t\omega^{-1})_n = (\omega k\omega^{-1})_{N+n} \tag{3.10}$$

for all $n\geq 1$. The derivative with respect to t corresponds to the equation of motion arising from $A_N(k)$. For each choice of N or k there will be a different equation of motion, and so the evolution operator ∂_t associated with $A_N(k)$ will be denoted $\partial_{N,k}$. The collection of these operators can be thought of as describing the evolution of the fields with respect to infinitely many independent time variables.

In this notation, (3.10) becomes

$$(\omega_{N,k}\omega^{-1})_n = (\omega k\omega^{-1})_{N+n} \tag{3.11}$$

for all $n\geq 1$, $N\geq 0$. In particular, choose n=1. Then, using (2.11b)

$$\partial_{N,k}\omega_1 = (\omega k\omega^{-1})_{N+1} \tag{3.12}$$

By (2.15), this implies

$$iq^{\alpha}_{N,k} = (\omega k\omega^{-1})^{\alpha}_{N+1} \qquad (3.13a)$$

$$iq^{\alpha *}_{N,k} = (\omega k\omega^{-1})^{-\alpha}_{N+1} \qquad (3.13b)$$

where $(\omega k\omega^{-1})^{\pm\alpha}$ is the coefficient of $e_{\pm\alpha}$ in $\omega k\omega^{-1}$. Equations (3.13) give the equation of motion corresponding to the pair A_x, $A_N(k)$. (One can check them directly from the zero curvature condition (2.6), using (2.4) and (3.9)). Consistency of (3.13a) and (3.13b) requires the restriction to the compact real form of $\underset{\sim}{g}$ [1], which means that k must be of the form

$$k^{i*} = -k^{i} \qquad (3.14a)$$

$$k^{a*} = -k^{-a} \qquad (3.14b)$$

(where $k = k^{i}h_{i}+k^{a}e_{a}+k^{-a}e_{-a}$).

4. POISSON BRACKET ALGEBRA.

The algebra of the evolution operators will now be investigated. This will allow the construction of the Poisson bracket algebra of the Hamiltonians for the equations of motion (3.13).

Recall equation (3.12), and act on both sides with the evolution operator $\partial_{M,j}$ ($M\geq 0$, $j \in \underset{\sim}{k}$) to obtain

$$\partial_{M,j}\partial_{N,k}\omega_1 = \partial_{M,j}(\omega k\omega^{-1})_{N+1}$$

$$= ([\omega_{M,j}\omega^{-1},\ \omega k\omega^{-1}])_{N+1}$$

$$= \Sigma_{p=0}^{N}[(\omega_{M,j}\omega^{-1})_{N+1-p}\ ,\ (\omega k\omega^{-1})_p]$$

$$= \Sigma_{p=0}^{N}[(\omega j\omega^{-1})_{M+N+1-p}\ ,\ (\omega k\omega^{-1})_p] \qquad (4.1)$$

(Use has been made of (3.11) and the identity

$$\partial_\mu(gXg^{-1}) = [g_\mu g^{-1}, gXg^{-1}] + g\partial_\mu Xg^{-1} \qquad \forall g \in G,\ X \in \underset{\sim}{g})$$

The same calculation with (M,j) and (N,k) interchanged leads to

$$[\partial_{N,k}\ ,\ \partial_{M,j}]\omega_1 = \Sigma_{q=0}^{N+M+1}[(\omega k\omega^{-1})_{N+M+1-q}\ ,\ (\omega j\omega^{-1})_q]$$

$$= ([\omega k\omega^{-1}, \omega j\omega^{-1}])_{N+M+1}$$

$$= (\omega[k,j]\omega^{-1})_{N+M+1}$$

$$= \partial_{N+M,[k,j]}\omega_1 \qquad (4.2)$$

using (3.12). In particular, (2.14) enables one to write

$$[\partial_{N,k}\ ,\ \partial_{M,j}]A_x^0 = \partial_{N+M,[k,j]}A_x^0 \qquad (4.3)$$

for all $N,M \geq 0$, $k,j \in \underset{\sim}{k}$.

Equation (4.3) states that the evolution operators form an algebra isomorphic to $\underset{\sim}{k}_R \otimes \mathbb{C}[\lambda]$. $\underset{\sim}{k}_R$ denotes the compact real form of $\underset{\sim}{k}$ (this distinction is necessary because of the consistency condition (3.14)) and $\mathbb{C}[\lambda]$ denotes the algebra of Laurent polynomials in positive powers of λ. The algebra defined by (4.3) can be thought of as "half" of a Kac-Moody algebra [4].

Now define the Poisson bracket between two functions A and B as

$$\{A,B\} = \sum_{\alpha}\int dz\left(\partial A/\partial q^{\alpha}(z).\partial B/\partial q^{\alpha *}(z)-\partial B/\partial q^{\alpha}(z).\partial A/\partial q^{\alpha *}(z)\right) \quad (4.4)$$

(arguments, delta functions etc. will subsequently be suppressed for clarity). The Hamiltonian $H_N(k)$ for the equation of motion (3.13) associated with $A_N(k)$ is defined by the relation

$$\partial_{N,k}A_x^0 = \{A_x^0,H_N(k)\} \quad (4.5)$$

(the Poisson bracket between an element of $\underset{\sim}{g}$, such as A_x^0, and a function, such as $H_N(k)$, is of course well defined). Definition (4.4) is equivalent to Hamilton's equations:

$$q^{\alpha}_{N,k} = \partial H_N(k)/\partial q^{\alpha *} \quad (4.6a)$$

$$q^{\alpha *}_{N,k} = -\partial H_N(k)/\partial q^{\alpha} \quad (4.6b)$$

Equations (4.3) and (4.5) can be used to rewrite the Jacobi identity

$$\{A_x^0\{H_N(k),H_M(j)\}\}+\{H_N(k)\{H_M(j),A_x^0\}\}+\{H_M(j)\{A_x^0,H_N(k)\}\} = 0 \tag{4.7}$$

in the form

$$\begin{aligned}\{A_x^0\{H_N(k),H_M(j)\}\} &= [\partial_{N,k}\ ,\ \partial_{M,j}]A_x^0 \\ &=\partial_{N+M,[k,j]}A_x^0 \\ &=\{A_x^0,H_{N+M}([k,j])\}\end{aligned} \tag{4.8}$$

which implies that

$$\{H_N(k),H_M(j)\} = H_{N+M}([k,j]) + C_{N,M}^{k,j} \tag{4.9}$$

where $C_{N,M}^{k,j}$ is a constant. Equation (4.9) states that the Poisson bracket algebra is the "half" Kac-Moody algebra with central extension. In fact, the central term can always be made to disappear by a suitable re-definition of the generators [5]. (In the present case, this is simply a reflection of the fact that the Hamiltonians are only defined up to a constant). For the case j=E, it is easy to check using the Jacobi identity that $C_{N,M}^{k,E}$ vanishes identically. In particular, this means that

$$\{H_N(k),H_2(E)\} = 0 \tag{4.10}$$

where $H_2(E)$ is the GNLS Hamiltonian. Therefore one can consider the entire collection of Hamiltonians $H_N(k)$ to be conserved quantities for the GNLS equation.

It only remains to find the explicit form of $H_N(k)$. First, put k=E and N=0 in (3.13):

$$iq^{\alpha}_{0,E} = [\omega_1, E]^{\alpha} = -q^{\alpha} \tag{4.11}$$

using (2.10b) and (2.14). It is then clear from (4.6) that

$$H_0(E) = i\int q^{\alpha}q^{\alpha *} \tag{4.12}$$

(summation implied). Now use (4.10), (4.5) to deduce

$$\partial_{N,k}\int q^{\alpha}q^{\alpha *} = 0 \tag{4.13}$$

It follows that

$$\int(q^{\alpha}q^{\alpha *}_{N,k} - q^{\alpha}_{N,k}q^{\alpha *}) = -2\int q^{\alpha}_{N,k}q^{\alpha *} = 2\int q^{\alpha *}_{N,k}q^{\alpha} \tag{4.14}$$

Next, use (3.12) to write

$$\int Tr(A^0_x\omega k\omega^{-1})_{N+1} = \int Tr(A^0_x\partial_{N,k}\omega_1)$$

$$= \int(-iq^{\alpha}q^{\alpha *}_{N,k} + iq^{\alpha}_{N,k}q^{\alpha *}) \tag{4.15}$$

Then use (4.14) and differentiate:

$$q^{\alpha}_{N,k} = -i/2\partial/\partial q^{\alpha *}\int Tr(A^0_x\omega k\omega^{-1})_{N+1} \tag{4.16a}$$

$$q^{\alpha *}_{N,k} = i/2\partial/\partial q^{\alpha}\int Tr(A^0_x\omega k\omega^{-1})_{N+1} \qquad (4.16b$$

Comparing these with (4.6), one can choose

$$H_N(k) = -i/2\int Tr(A_x^0 \omega k \omega^{-1})_{N+1} \tag{4.17}$$

5. THE GNLS HIERARCHY.

It is clear from the construction of ω in Section 2 that the operators $\partial_{N,k}$ give rise, in general, to non-local equations of motion (with non-local Hamiltonians). What is, perhaps, rather surprising is that for k=E the equations of motion (the GNLS hierarchy) are all local. To show this, the objects $A_N(E)$, $H_N(E)$ and $\partial_{N,E}$ will here be reconstructed in terms of local quantities.

Consider the gauge transformation w which takes A_x into $\underset{\sim}{k}$:

$$A_x \rightarrow \tilde{a}_x = w^{-1} A_x w + w^{-1} w_x = \lambda E + \Sigma_{n=1}^{\infty} \lambda^{-n} \tilde{a}_x^n \tag{5.1}$$

It was shown in Section 2 that this is a local gauge transformation. Now, as in Section 3, one wishes to find $\tilde{a}_t$ such that the zero curvature condition

$$\partial_x \tilde{a}_t - \partial_t \tilde{a}_x + [\tilde{a}_x, \tilde{a}_t] = 0 \tag{5.2}$$

is satisfied, where $\tilde{a}_t$ has the general form

$$\tilde{a}_t = \lambda^N \tilde{a}_t^{-N} + \ldots + \tilde{a}_t^0 + \lambda^{-1} \tilde{a}_t^1 + \ldots \tag{5.3}$$

Substitute (5.1) and (5.3) into (5.2), and equate coefficients of powers of λ:

$$\lambda^{N+1}: \quad [E,\tilde{a}_t^{-N}] = 0 \qquad \text{i.e.} \quad \tilde{a}_t^{-N} \in \underset{\sim}{k} \tag{5.4}$$

$$\lambda^{N}: \quad \partial_x \tilde{a}_t^{-N} + [E,\tilde{a}_t^{-(N-1)}] = 0 \tag{5.5}$$

i.e. $\tilde{a}_t^{-(N-1)} \in \underset{\sim}{k}$ and $\tilde{a}_t^{-N}$ is a constant. Choose $\tilde{a}_t^{-N}=E$.

$$\lambda^{N-1}: \quad \partial_x \tilde{a}_t^{-(N-1)} + [E,\tilde{a}_t^{-(N-2)}] + [\tilde{a}_x^{1},\tilde{a}_t^{-N}] = 0 \tag{5.6}$$

Again, split this into parts in $\underset{\sim}{m}$ and $\underset{\sim}{k}$ to find $\tilde{a}_t^{-(N-2)} \in \underset{\sim}{k}$ and

$$\partial_x \tilde{a}_t^{-(N-1)} = [\tilde{a}_t^{-N},\tilde{a}_x^{1}] \tag{5.7}$$

Since $\tilde{a}_t^{-N}=E$, and $\tilde{a}_x^{1} \in \underset{\sim}{k}$, this becomes

$$\tilde{a}_t^{-(N-1)} = \text{constant} \tag{5.8}$$

Choose $\tilde{a}_t^{-(N-1)}=0$, and continue in the same fashion. one finds that $\tilde{a}_t$ can be chosen to have the form

$$\tilde{a}_t = \lambda^N E + \Sigma_{n=1}^{\infty} \lambda^{-n} \tilde{a}_t^{n} \tag{5.9}$$

Now invert the gauge transformation:

$$A_t = w\tilde{a}_t w^{-1} - w_t w^{-1} \tag{5.10}$$

and equate positive powers of λ to obtain

$$A_t = \Sigma_{n=0}^{N}\lambda^n(wEw^{-1})_{N-n} \tag{5.11}$$

This has leading term $\lambda^N E$, and is equal to $A_N(E)$ as given by (3.9) with k=E (and the constants of integration set to zero). It immediately follows that

$$\omega E\omega^{-1} = wEw^{-1} \tag{5.12}$$

to all orders. One can deduce from this that the equations of motion (3.13) and Hamiltonians (4.17) become local for k=E. Notice, incidentally, that the equation of motion cannot be read off from the coefficient of λ^{-1} in (5.10), since $\tilde{a}_t^1$ is non-zero. One can, however, obtain it from the zero curvature condition:

$$\begin{aligned}
\partial_t A_x^0 &= \partial_x A_t^0 + [A_x^0, A_t^0] \\
&= ([w_x w^{-1}, wEw^{-1}])_N + [A_x^0, (wEw^{-1})_N] \\
&= -([\lambda E, wEw^{-1}])_N \qquad \text{(since } [\tilde{a}_x, E] = 0) \\
&= -[E, (wEw^{-1})_{N+1}]
\end{aligned} \tag{5.13}$$

Finally, the Hamiltonians $H_N(E)$ will be calculated for N=0,1,2. One uses (2.23), (2.24), (2.25) to obtain

$$\mathrm{Tr}(A_x^0 wEw^{-1})_1 = \mathrm{Tr}(A_x^0[w_1, E]) = \mathrm{Tr}(A_X^0 A_X^0) \tag{5.14}$$

$$\mathrm{Tr}(A_x^0 wEw^{-1})_2 = \mathrm{Tr}(A_x^0\{[w_2,E] + 1/2[w_1[w_1,E]]\})$$

$$= \mathrm{Tr}(A_x^0[w_2,E]) \quad (\text{since } [w_1[w_1,E]] \in \underset{\sim}{k})$$

$$= \mathrm{Tr}(E[\partial_x A_x^0, A_x^0]) \tag{5.15}$$

$$\mathrm{Tr}(A_x^0 wEw^{-1})_3 = \mathrm{Tr}(A_x^0([w_3,E] + 1/6[w_1[w_1[w_1,E]]]))$$

(only terms in $\underset{\sim}{m}$ contribute)

$$= -\mathrm{Tr}(A_x^0 \partial_{xx} A_x^0) - 1/2\mathrm{Tr}([A_x^0[A_x^0,E]]^2) \tag{5.16}$$

One can work these out explicitly in terms of the fields q^α, $q^{\alpha*}$ (see Appendix I) to find

$$H_0(E) = i\int q^\alpha q^{\alpha*} \tag{5.17}$$

$$H_1(E) = 1/2\int q_x^\alpha q^{\alpha*} - q^\alpha q_x^{\alpha*} \tag{5.18}$$

$$H_2(E) = i\int q_x^\alpha q_x^{\alpha*} + q^{\alpha*} q^\beta q^\gamma q^{\delta*} R^\alpha_{\beta\gamma-\delta} \tag{5.19}$$

(integration by parts has been used in (5.19)). (5.17) and (5.18) are straightforward generalizations of the "particle number" and "momentum" of the NLS equation [2]. (5.19) gives the Hamiltonian of the GNLS equation.

One can also check the expressions for $A_N(E)$ and $\partial_{N,E}$. For example, from (5.13)

$$\partial_{2,E}A_x^0 = -([E,wEw^{-1}])_3$$

$$= [E,\ \partial_{xx}A_x^0 - 1/2[w_1[w_1,A_x^0]]] \qquad (5.20)$$

In terms of the fields this becomes

$$iq^\alpha_{\ 2,E} = q^\alpha_{xx} - q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta} \qquad (5.21a)$$

$$-iq^{\alpha *}_{\ 2,E} = q^{\alpha *}_{xx} - q^{\beta *} q^{\gamma *} q^\delta R^{-\alpha}_{-\beta-\gamma\delta} \qquad (5.21b)$$

as expected. The calculation of $A_2(E)$ is as follows:

$$A_2(E) = \Sigma^2_{n=0}\lambda^n(wEw^{-1})_{N-n}$$

$$= \lambda^2 E + \lambda[w_1,E] + [w_2,E] + 1/2[w_1[w_1,E]]$$

$$= \lambda^2 E + \lambda A_x^0 + [E,\partial_x A_x^0] + 1/2[A_x^0[A_x^0,E]] \qquad (5.22)$$

(using (2.23), (2.24)). This is in agreement with (1.6b).

Of course, equation (5.11) ensures that the same results would be obtained if ω were used instead of w, although the calculation is more complicated.

6. DISCUSSION.

The GNLS equation has two important special cases. As was mentioned earlier, the familiar non-linear Schrödinger equation corresponds to $\underset{\sim}{g}$=su(2). In that case, $\underset{\sim}{k}$ is the one dimensional Cartan subalgebra, so that any element of $\underset{\sim}{k}$ is a scalar multiple of E. Consequently only the local series of charges exists. The GNLS equation associated with $SU(n+1)/(U(1) \times SU(n))$ is known as the vector non-linear Schrödinger equation, and has arisen (like the NLS equation) in non-linear optics [6]. Non-local charges will exist for $n \geq 2$. It would be interesting to find out whether such quantities could have any physical significance.

A major step in the construction of $H_N(k)$ was to find a general form for $A_N(k)$. For k=E, the same expression can, in fact, be found using the P-operator method of Olive and Turok [3] (the P-operator in the present case is the Casimir operator for $\underset{\sim}{g} \otimes \underset{\sim}{g}$ [1]) although the conditions they assume no longer hold (i.e. E is not regular).

As a generalization of the system considered here, one could begin with a trivial solution of the zero curvature condition:

$$a_x = \lambda^p \tilde{A} \qquad (6.1a)$$

$$a_t = \lambda^N A \qquad (6.1b)$$

where $\tilde{A}$, A are constants and $[\tilde{A},A]=0$. One then finds A_x, A_t as series in positive powers of λ using the inverse gauge transformation

$$A_\mu = \omega a_\mu \omega^{-1} - \omega_\mu \omega^{-1} \qquad (6.2)$$

Those coefficients ω_n which remain undetermined by the requirement that (6.2) be consistent can be considered as dynamical fields (for the GNLS case this was $\tilde{\omega}_1^m$). This will be discussed further in a subsequent paper. The evolution operators will obey the same "half" Kac-Moody algebra, but the precise form of the Hamiltonians will depend on the structure of A_x. It is anticipated that a generalization of the P-operator method will be applicable.

APPENDIX.

Some results are given here concerning Lie algebras and symmetric spaces. Further details can be found in, e.g., [7].

The Cartan-Weyl basis $\{h_i, e_r : h_i \in \underset{\sim}{h}, r \in \Phi\}$ of a complex semi-simple Lie algebra $\underset{\sim}{g}$, with Cartan subalgebra $\underset{\sim}{h}$, satisfies the following relations (where Φ is the set of roots and $r \in \Phi$ can be positive or negative):

$$[h_i, h_j] = 0 \tag{A.1a}$$

$$[h_i, e_r] = r_i e_r \tag{A.1b}$$

(If $H = H^i h_i \in \underset{\sim}{h}$, where summation over i is implied, then

$$[H, e_r] = H^i r_i e_r \equiv H.re_r \tag{A.1c}$$

The dot is used to indicate summation over Cartan subalgebra indices).

$$[e_r, e_{-r}] = r.h \tag{A.1d}$$

If $r \neq -s$ then

$$[e_r, e_s] = N_{r,s} e_{r+s} \tag{A.1e}$$

where $N_{r,s}=0$ if r+s is not a root. One can check the useful identities:

$$N_{r,s} = -N_{-r,-s} = N_{-s,-r} = N_{s,-r-s} \tag{A.2}$$

The basis is scaled so that

$$\mathrm{Tr}(h_i h_j) = \delta_{ij} \tag{A.3a}$$

$$\mathrm{Tr}(e_r e_{-s}) = \delta_{rs} \tag{A.3b}$$

$$\mathrm{Tr}(h_i e_r) = 0 \tag{A.3c}$$

From now on, r,s,... will denote only positive roots.

For any element $A \in \underset{\sim}{g}$ define the "centralizer" C(A) of A by

$$C(A) = \{B \in \underset{\sim}{g} : [A,B]=0\} \tag{A.4}$$

An element $H \in \underset{\sim}{h}$ is called regular if $C(H)=\underset{\sim}{h}$.

Let $E \in \underset{\sim}{h}$ be an element with the property that for any (positive) root r, E.r is either zero or takes a constant value κ. (Such an element does not always exist - for example E_8 does not possess one). Now define the set θ^+ of roots which satisfy

$$E.\alpha = \kappa \tag{A.5}$$

for all $\alpha \in \theta^+$. Denoting by Φ^+ the set of positive roots, and defining $\bar{\theta}^+ \equiv \Phi^+ - \theta^+$, then

$$E.a = 0 \tag{A.6}$$

for all $a \in \bar{\theta}^+$. The Greek letters $\alpha, \beta, \gamma, \ldots$ will always denote elements of θ^+, and the Latin letters a,b,c,... will denote elements of $\bar{\theta}^+$.

C(E) is a subalgebra spanned by $\{h_i, e_{\pm a}: h_i \in \underset{\sim}{h}, a \in \bar{\theta}^+\}$, which will be denoted by $\underset{\sim}{k}$. Then

$$\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m} \tag{A.7}$$

where $\underset{\sim}{m}$, the orthogonal complement of $\underset{\sim}{k}$, is a subspace spanned by $\{e_{\pm\alpha}: \alpha \in \theta^+\}$. Notice that $[E,A] \in \underset{\sim}{m}$ for any element A $\underset{\sim}{g}$. Also

$$[E[E,A]] = \kappa^2 A^{\underset{\sim}{m}} \tag{A.8}$$

where $A^{\underset{\sim}{m}}$ is the component of A in $\underset{\sim}{m}$. The Jacobi identity implies the useful special cases:

$$[[E,m]k] = [E[m,k]] \qquad \text{for all } m \in \underset{\sim}{m},\ k \in \underset{\sim}{k} \tag{A.9}$$

$$[[E,m_1]m_2] = [[E,m_2]m_1] \quad \text{for all } m_1,\ m_2 \in \underset{\sim}{m} \tag{A.10}$$

From the definition of E one deduces the following:

$$[e_\alpha, e_\beta] = [e_{-\alpha}, e_{-\beta}] = 0 \tag{A.11}$$

$$[e_\alpha, e_{-\beta}] \in \underset{\sim}{k} \tag{A.12}$$

$$\alpha \pm a \in \theta^+ \text{ (if it is a root)} \tag{A.13}$$

Then

$$[\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \qquad [\underset{\sim}{k},\underset{\sim}{m}] \subset \underset{\sim}{m} \qquad [\underset{\sim}{m},\underset{\sim}{m}] \subset \underset{\sim}{k} \tag{A.14}$$

i.e. $\underset{\sim}{g}$ is a "symmetric algebra" and G/K is a symmetric space. The curvature tensor is defined as

$$R_{\pm\alpha\pm\beta\pm\gamma} = [e_{\pm\alpha}[e_{\pm\beta},e_{\pm\gamma}]] \tag{A.15}$$

The identity (A.11) implies

$$R_{\alpha\beta\gamma} = R_{\alpha-\beta-\gamma} = 0 \tag{A.16}$$

while (A.12), (A.13) give

$$R^{-\delta}_{\alpha\beta-\gamma} = 0 \tag{A.17}$$

etc. In fact, the symmetric spaces constructed in the way described above are "Hermitian", and the curvature tensor satisfies

$$(R^{\alpha}_{\beta\gamma-\delta})^* = R^{-\alpha}_{-\beta-\gamma\delta} \tag{A.18}$$

Finally, it is useful to give the commutator for two general elements of $\underset{\sim}{g}$. Writing the components as

$$A = A.h + A^a e_a + A^{-a} e_{-a} + A^{\alpha} e_{\alpha} + A^{-\alpha} e_{-\alpha} \tag{A.19}$$

then

$$[A,B] = (A^{a}B^{-a} - A^{-a}B^{a})a.h + (A^{\alpha}B^{-\alpha} - A^{-\alpha}B^{\alpha})\alpha.h$$

$$+ (A.aB^{a} - A^{a}B.a + A^{b}B^{a-b}N_{-a,b} + A^{-b}B^{a+b}N_{-a,-b}$$

$$+ A^{\alpha}B^{-\alpha+a}N_{-a,\alpha} + A^{-\alpha}B^{\alpha+a}N_{-a,-\alpha})e_{a}$$

$$+ (A^{-a}B.a - A.aB^{-a} + A^{b}B^{-a-b}N_{a,b} + A^{-b}B^{-a+b}N_{a,-b}$$

$$+ A^{\alpha}B^{-\alpha-a}N_{a,\alpha} + A^{-\alpha}B^{\alpha-a}N_{a,-\alpha})e_{-a}$$

$$+ (A.\alpha B^{\alpha} - A^{\alpha}B.\alpha + A^{\alpha-\beta}B^{\beta}N_{\alpha,-\beta} + A^{\beta}B^{\alpha-\beta}N_{-\alpha,\beta})e_{\alpha}$$

$$+ (A^{-\alpha}B.\alpha - A.\alpha B^{-\alpha} + A^{\beta-\alpha}B^{-\beta}N_{-\alpha,\beta} + A^{-\beta}B^{\beta-\alpha}N_{\alpha,-\beta})e_{-\alpha}$$

(A.20)

REFERENCES.

[1] Fordy, A.P., Kulish, P.P.: Nonlinear Schrödinger equations and simple Lie algebras. Commun. Math. Phys. 89, 427-443 (1983)

[2] Faddeev, L.D. In: Recent advances in field theory and statistical mechanics, Les Houches, Session 39, Elsevier Sci. Publ.563-608 (1984)

[3] Olive, D., Turok, N.: Local conserved densities and zero curvature conditions for Toda lattice field theories. Nucl. Phys.B257[FS14], 277 (1985)

[4] Dolan, L.: Kac-Moody algebras and exact solvability in hadronic physics. Phys. Rep. 109 No. 1, 1-94 (1984)

[5] Goddard, P., Olive, D.: Kac-Moody and Virasoro algebras in relation to quantum physics. To appear.

[6] Manakov, S.V.: On the theory of two dimensional stationary self-focusing of electromagnetic waves. Sov. Phys. JETP 38 No.2 248-253 (1974)

[7] Helgason, S.: Differential geometry, Lie groups and symmetric spaces, 2nd ed. New York: Academic Press (1978)

CHAPTER 3: KAC-MOODY SYMMETRY OF GENERALIZED NON-LINEAR SCHRÖDINGER EQUATIONS

1. INTRODUCTION

This is a continuation of the work presented in [1], in which it was shown how to construct conserved quantities for the generalized non-linear Schrödinger (GNLS) equation of Fordy and Kulish [2]:

$$iq^{\alpha}_{t} = q^{\alpha}_{xx} \pm q^{\beta}q^{\gamma}q^{\delta *}R^{\alpha}_{\beta\gamma-\delta} \tag{1.1}$$

(summation is implied over repeated indices) which is associated with a Lie algebra $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$. $q(x,t)$ is a matrix field in 1+1 dimensions whose components lie in $\underset{\sim}{m}$, and $\underset{\sim}{k}$ is the centralizer of a special Cartan subalgebra element E satisfying the property

$$[E,e_{\alpha}] = -ie_{\alpha} \tag{1.2}$$

for all $e_{\alpha} \in \underset{\sim}{m}$ (where α is positive). This means that the algebra $\underset{\sim}{g}$ is "symmetric", i.e.

$$[\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \qquad [\underset{\sim}{k},\underset{\sim}{m}] \subset \underset{\sim}{m} \qquad [\underset{\sim}{m},\underset{\sim}{m}] \subset \underset{\sim}{k} \tag{1.3}$$

The curvature tensor R has components in $\underset{\sim}{m}$ defined by

$$e_{\alpha}R^{\alpha}_{\beta\gamma-\delta} = [e_{\beta}[e_{\gamma},e_{-\delta}]] \tag{1.4}$$

Equation (1.1) can be written as a zero-curvature condition

$$\partial_{x}A_{t} - \partial_{t}A_{x} + [A_{x},A_{t}] = 0 \tag{1.5}$$

where

$$A_x = \lambda E + A_x^0 \tag{1.6a}$$

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]] \tag{1.6b}$$

and

$$A_x^0 = -q^\alpha e_\alpha - q^{-\alpha} e_{-\alpha} \tag{1.7}$$

The component form of (1.5) is

$$iq_t^\alpha = q_{xx}^\alpha + q^\beta q^\gamma q^{-\delta} R^\alpha_{\beta\gamma-\delta} \tag{1.8a}$$

$$-iq_t^{-\alpha} = q_{xx}^{-\alpha} + q^{-\beta} q^{-\gamma} q^\delta R^{-\alpha}_{-\beta-\gamma\delta} \tag{1.8b}$$

The choices $q^{-\alpha} = \pm q^{\alpha *}$ correspond to the restriction to the non-compact (+) or compact (-) real forms of $\underset{\sim}{g}$, and lead to equation (1.1) with a plus or minus sign.

One can find other values of A_t as a polynomial in λ such that the new equation of motion (1.5), with A_x given by (1.6a), is still independent of λ. Each such A_t is associated, via (1.5), with an evolution operator ∂_t. It was shown in [1] that when A_t is a polynomial in positive powers only, the collection of evolution operators can be labelled $\partial_{N,k}$, where $k \in \underset{\sim}{k}$ and N is a positive integer, and that they have the commutation relation

$$[\partial_{M,j}, \partial_{N,k}] = \partial_{M+N,[j,k]} \qquad \forall M,N \geq 0; j,k \in \underset{\sim}{k} \tag{1.9}$$

In this paper, the case will be considered when A_t is a polynomial in negative powers of λ. This will lead to the construction of evolution operators $\partial_{-N,k}$ such that

$$[\partial_{m,j},\partial_{n,k}] = \partial_{m+n,[j,k]} \qquad \forall m,n \in Z;\ j,k \in \underset{\sim}{k} \tag{1.10}$$

The complete collection of operators $\partial_{\pm N,k}$ provides a realization of the Kac-Moody algebra $\underset{\sim}{k} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$, where $\mathbb{C}[\lambda,\lambda^{-1}]$ is the algebra of Laurent polynomials in the complex variable λ. The parameters $(\pm N,k)$ are thought of as infinitely many independent "time" variables.

In [1] it was shown how to construct a group element of the form

$$\omega = \exp\Sigma_{n=1}^{\infty}\lambda^{-n}\omega_n \tag{1.11}$$

defined by

$$\lambda\omega E\omega^{-1} - \omega_x\omega^{-1} = \lambda E + A_x^0 \tag{1.12}$$

Under the gauge transformation

$$A_x \to \omega^{-1}A_x\omega + \omega^{-1}\omega_x = \lambda E \tag{1.13a}$$

$$A_t \to \omega^{-1}A_t\omega + \omega^{-1}\omega_t = a_t \tag{1.13b}$$

where A_t is unknown, the zero curvature condition (1.5) becomes

$$\partial_x a_t + [\lambda E, a_t] = 0 \tag{1.14}$$

This equation can be satisfied by choosing

$$a_t = \lambda^N k \tag{1.15}$$

where N is a positive integer and $k \in \underset{\sim}{k}$ is constant. The transformation (1.13b) is inverted to obtain

$$A_t = \lambda^N \omega k \omega^{-1} - \omega_t \omega^{-1} \tag{1.16}$$

If A_t is chosen to have no negative powers of λ, then it is determined uniquely by (1.16) as the positive power part of $\lambda^N \omega k \omega^{-1}$, while the action of ∂_t on ω is determined by the negative power part. A_t and ∂_t defined in this way are denoted $A_N(k)$, $\partial_{N,k}$. Equating coefficients of powers of λ in (1.16) one obtains

$$A_N(k) = \Sigma_{n=0}^N \lambda^n (\omega k \omega^{-1})_{N-n} \tag{1.17a}$$

$$(\omega_{N,k} \omega^{-1})_n = (\omega k \omega^{-1})_{N+n} \tag{1.17b}$$

where $(\ldots)_n$ denotes the coefficient of λ^{-n}. The relation (1.9) follows from the definition (1.17b).

Now suppose that one chooses

$$a_t = \lambda^{-N} k \tag{1.18}$$

as a solution to (1.14). Then the inverse gauge transformation (1.13b) gives

$$A_{-N}(k) = \lambda^{-N}\omega k\omega^{-1} - \omega_{-N,k}\omega^{-1} \tag{1.19}$$

which does not determine $A_{-N}(k)$. For example, if $N>1$ then the coefficient of λ^{-1} in (1.19) is

$$A^{1}_{-N}(k) = -\partial_{-N,k}\omega_1 \tag{1.20}$$

One can think of (1.19) as defining the action of $\partial_{-N,k}$ on ω in terms of the as yet undetermined $A_{-N}(k)$. To find $A_{-N}(k)$ as a polynomial in negative powers of λ, one would like to have an equation like (1.19) in which ω is replaced by a group element which contains only non-negative powers of λ, i.e. one would like to find $\tilde{\omega}$ of the form

$$\tilde{\omega} = \exp\Sigma^{\infty}_{n=0}\lambda^{n}\tilde{\omega}_n \tag{1.21}$$

such that one can perform the transformation

$$A_x \to \tilde{\omega}^{-1}A_x\tilde{\omega} + \tilde{\omega}^{-1}\tilde{\omega}_x = \lambda E \tag{1.22a}$$

$$A_t \to \tilde{\omega}^{-1}A_t\tilde{\omega} + \tilde{\omega}^{-1}\tilde{\omega}_t = a_t \tag{1.22b}$$

Then one can again consider the solutions $a_t = \lambda^{\pm N}k$ for (1.14). For the case $\lambda^{-N}k$, the inverse transformation (1.22b) gives

$$A_{-N}(k) = \lambda^{-N}\tilde{\omega}k\tilde{\omega}^{-1} - \tilde{\omega}_{-N,k}\tilde{\omega}^{-1} \tag{1.23}$$

which enables one to obtain $A_{-N}(k)$ and the action of $\partial_{-N,k}$ on $\tilde{\omega}$, by equating coefficients of powers of λ. The case $\lambda^N k$ defines the action of $\partial_{N,k}$ on $\tilde{\omega}$ in terms of $A_N(k)$ (1.17a). To construct $\tilde{\omega}$, one writes it in the form

$$\tilde{\omega} = \phi\Omega \qquad (1.24)$$

where ϕ is independent of λ, and

$$\Omega = \exp\Sigma_{n=1}^{\infty}\lambda^n\Omega_n \qquad (1.25)$$

It will be shown in Section 2 that equation (1.22a) determines Ω to all orders in terms of the auxiliary field ϕ. In Section 3 the commutation relations of the evolution operators $\partial_{\pm N,k}$ will be investigated, which will show them to form a realization of a Kac-Moody algebra. The class of operators can be extended by allowing the algebra element to be an arbitrary element of $\underset{\sim}{g}$, rather than just of $\underset{\sim}{k}$. In Section 4 the Hamiltonians for the operators $\partial_{\pm N,g}$ are considered. Their Poisson bracket algebra provides a realization of the Kac-Moody algebra $\underset{\sim}{g} \otimes \mathbb{C}[\lambda,\lambda^{-1}] \oplus \mathbb{C}c$. In Section 5 it is shown that the formal sum of Hamiltonians for the operators $\partial_{\pm N,e_\alpha}$ can be used to linearize the system. The interpretation of these operators is discussed in Section 6.

2. CONSTRUCTION OF $\tilde{\omega}$

Let $\tilde{\omega}$ be an element of the Lie group G, of the form

$$\tilde{\omega} = \phi\Omega \tag{2.1}$$

where ϕ is independent of λ, and

$$\Omega = \exp \Sigma_{n=1}^{\infty} \lambda^n \Omega_n \tag{2.2}$$

Now fix $\tilde{\omega}$ by choosing

$$\lambda\tilde{\omega}E\tilde{\omega}^{-1} - \tilde{\omega}_x\tilde{\omega}^{-1} = A_x \tag{2.3}$$

where A_x is given by (1.6a); i.e.

$$\lambda E + A_x^0 = \lambda\phi\Omega E\Omega^{-1}\phi^{-1} - \phi\Omega_x\Omega^{-1}\phi^{-1} - \phi_x\phi^{-1} \tag{2.4}$$

Equating coefficients of powers of λ^0, one can see that

$$A_x^0 = -\phi_x\phi^{-1} \tag{2.5}$$

Notice that ϕ is the group element which arises in the transformation between the GNLS system and the generalized Heisenberg ferromagnet [2]. (This will be explained more fully in Section 6). The λ-dependent part of (2.4) becomes

$$\lambda\Omega E\Omega^{-1} - \Omega_x\Omega^{-1} = \lambda\phi^{-1}E\phi \tag{2.6}$$

Now, by expanding (2.2) as a power series in λ, one can obtain the identities

$$(\Omega E\Omega^{-1})_n = \Sigma_{r=1}^{n}(r!)^{-1}\Sigma_{k_i:\Sigma k_i=n}[\Omega_{k_1}[\Omega_{k_2}[\ldots[\Omega_{k_r},E]\ldots]]] \tag{2.7 a}$$

$$(\Omega_x\Omega^{-1})_n =$$

$$\Sigma_{r=1}^{n}(r!)^{-1}\Sigma_{k_i:\Sigma k_i=n}[\Omega_{k_1}[\Omega_{k_2}[\ldots[\Omega_{k_{r-1}},\partial_x\Omega_{k_r}]\ldots]]] \tag{2.7b}$$

where $(\ldots)_n$ denotes the coefficient of λ^n. Use these to equate coefficients of λ^n in (2.6):

λ^1: $\quad \partial_x\Omega_1 = E - \phi^{-1}E\phi$

i.e. $\quad \Omega_1 = xE - \partial^{-1}(\phi^{-1}E\phi)$ (2.8a)

λ^2: $\quad \partial_x\Omega_2 + 1/2[\Omega_1,\partial_x\Omega_1] = [\Omega_1,E]$

i.e. $\quad \Omega_2 = 1/2\partial^{-1}([E,\partial^{-1}(\phi^{-1}E\phi)] + x[E,\phi^{-1}E\phi]$

$\quad + [\phi^{-1}E\phi,\partial^{-1}(\phi^{-1}E\phi)])$ (2.8b)

and so on. In general one has

$$(\Omega_x\Omega^{-1})_n = (\Omega E\Omega^{-1})_{n-1} \tag{2.9}$$

for all $n>1$, and so $\partial_x\Omega_n$ is determined in terms of $\Omega_{m<n}$. In this way, Ω is determined to all orders non-locally in terms of ϕ.

3. THE EVOLUTION OPERATORS

Recall the zero curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0 \qquad (3.1)$$

where A_x is given by (1.6a) and A_t is unknown. Consider the gauge transformation

$$A_x \to \omega^{-1} A_x \omega + \omega^{-1} \omega_x = \lambda E \qquad (3.2a)$$

$$A_t \to \omega^{-1} A_t \omega + \omega^{-1} \omega_t = a_t \qquad (3.2b)$$

where ω is the group element defined by (1.12), of the form

$\omega = \exp\Sigma_{n=1}^{\infty} \lambda^{-n} \omega_n$. Under the transformation (3.2), the zero curvature condition (3.1) becomes

$$\partial_x a_t + [\lambda E, a_t] = 0 \qquad (3.3)$$

One can choose the solutions

$$a_t = \lambda^{\pm N} k \qquad (3.4)$$

for (3.3) (where N is a positive integer). Then (3.2b) can be inverted to obtain

$$A_N(k) = \lambda^N \omega k \omega^{-1} - \omega_{N,k} \omega^{-1} \qquad (3.5a)$$

$$\omega_{-N,k}\omega^{-1} = \lambda^{-N}\omega k\omega^{-1} - A_{-N}(k) \tag{3.5b}$$

where $A_N(k)$ is chosen to be a polynomial in non-negative powers of λ, while $A_{-N}(k)$ is a polynomial in negative powers. Equation (3.5a) defines $A_N(k)$ and the action of $\partial_{N,k}$ on ω, while (3.5b) is regarded as defining the action of $\partial_{-N,k}$ on ω in terms of $A_{-N}(k)$, which is still undetermined.

Now consider the gauge transformation (3.2) with ω replaced by $\tilde{\omega}$ as constructed in Section 2. Then, from the definition (2.3),

$$A_x \to \tilde{\omega}^{-1}A_x\tilde{\omega} + \tilde{\omega}^{-1}\tilde{\omega}_x = \lambda E \tag{3.6a}$$

$$A_t \to \tilde{\omega}^{-1}A_t\tilde{\omega} + \tilde{\omega}^{-1}\tilde{\omega}_t = a_t \tag{3.6b}$$

where A_t is considered unknown. The zero curvature condition again takes the form (3.3), and the solutions (3.4) can be used to invert (3.6b) to give

$$A_{-N}(k) = \lambda^{-N}\tilde{\omega}k\tilde{\omega}^{-1} - \tilde{\omega}_{-N,k}\tilde{\omega}^{-1} \tag{3.7a}$$

$$\tilde{\omega}_{N,k}\tilde{\omega}^{-1} = \lambda^{N}\tilde{\omega}k\tilde{\omega}^{-1} - A_N(k) \tag{3.7b}$$

Equation (3.7a) defines $A_{-N}(k)$ as the negative power part of $\lambda^{-N}\tilde{\omega}k\tilde{\omega}^{-1}$, and defines $\tilde{\omega}_{-N,k}\tilde{\omega}^{-1}$ as the positive power part. Equation (3.7b) defines the action of $\partial_{N,k}$ on $\tilde{\omega}$ in terms of $A_N(k)$, which was defined by the positive power part of (3.5a). Explicitly, one has

$$A_N(k) = \Sigma_{n=0}^{N}\lambda^{n}(\omega k\omega^{-1})_{N-n} = \Sigma_{n=0}^{N}\lambda^{n}A_N^{n}(k) \quad (3.8a)$$

$$(\omega_{N,k}\omega^{-1})_n = (\omega k\omega^{-1})_{N+n} \quad (3.8b)$$

(from (3.5a)), where $(f(\omega))_n$ denotes the coefficient of λ^{-n} in $f(\omega)$,

$$A_{-N}(k) = \Sigma_{n=1}^{N}\lambda^{-n}\phi(\Omega k\Omega^{-1})_{N-n}\phi^{-1} = \Sigma_{n=1}^{N}\lambda^{-n}A_{-N}^{n}(k) \quad (3.9a)$$

$$(\Omega_{-N,k}\Omega^{-1})_n = (\Omega k\Omega^{-1})_{N+n} \quad (3.9b)$$

for N>0 (from (3.7a), using (2.1)), where $(f(\Omega))_n$ denotes the coefficient of λ^{n} in $f(\Omega)$,

$$\phi_{N,k}\phi^{-1} = -A_N^{0}(k) = -(\omega k\omega^{-1})_N \qquad (\forall N > 0) \quad (3.10a)$$

$$\phi_{0,k}\phi^{-1} = \phi k\phi^{-1} - k \quad (3.10b)$$

(from the coefficient of λ^{0} in (3.7b), using (2.1) and (3.8a)),

$$\phi_{-N,k}\phi^{-1} = \phi(\Omega k\Omega^{-1})_N\phi^{-1} \qquad (\forall N > 0) \quad (3.11)$$

(from the coefficient of λ^{0} in (3.7a)). Lastly, (3.5b) and (3.7b) become

$$\omega_{-N,k}\omega^{-1} = \lambda^{-N}\omega k\omega^{-1} - \Sigma_{n=1}^{N}\lambda^{-n}\phi(\Omega k\Omega^{-1})_{N-n}\phi^{-1} \quad (\forall N > 0) \quad (3.12a)$$

$$\Omega_{N,k}\Omega^{-1} = \lambda^{N}\Omega k\Omega^{-1} - \Sigma_{n=1}^{N}\lambda^{n}\phi^{-1}(\omega k\omega^{-1})_{N-n}\phi \qquad (\forall N > 0) \quad (3.12b)$$

$$\Omega_{0,k}\Omega^{-1} = \Omega k\Omega^{-1} - k \tag{3.12c}$$

(using (3.9a), (3.8a) and (3.10)).

Notice that (3.8b) implies

$$(\omega_{1,E}\omega^{-1})_n = (\omega E\omega^{-1})_{n+1} = (\omega_x\omega^{-1})_n \tag{3.13}$$

(by (1.12)), i.e.

$$\partial_{1,E} = \partial_x \tag{3.14}$$

and so, by (3.10a),

$$\psi_x\psi^{-1} = -(\omega E\omega^{-1})_1 = -A_x^0 \tag{3.15}$$

(using (1.12) again). This is consistent with (2.5). Now, ω satisfies identities like (2.7), where $(\ldots)_n$ is taken to denote the coefficient of λ^{-n}, so that (from (3.15))

$$\partial_{N,k}A_x^0 = [\partial_{N,k}\omega_1, E]$$

$$= [(\omega_{N,k}\omega^{-1})_1, E] \qquad \text{(using (2.7b))}$$

$$= [(\omega k\omega^{-1})_{N+1}, E] \qquad \text{(from (3.8b))} \tag{3.16a}$$

and

$$\partial_{-N,k}A_x^0 = [\partial_{-N,k}\omega_1, E]$$

$$= [(\omega_{N,k}\omega^{-1})_1, E] \qquad \text{(using (2.7b))}$$

$$= [(\omega k\omega^{-1})_{N+1}, E] \qquad \text{(from (3.8b))} \qquad (3.16a)$$

and

$$\partial_{-N,k}A^0_x = [\partial_{-N,k}\omega_1, E]$$

$$= [E, A^1_{-N}(k)] \qquad \text{(from the coefficient of } \lambda^{-1} \text{ in (3.5b))}$$

$$\text{i.e.} \quad -\partial_{-N,k}(\phi_x\phi^{-1}) = [E, \phi(\Omega k\Omega^{-1})_{N-1}\phi^{-1}] \qquad (3.16b)$$

(by (3.9a) and (2.5)). Equations (3.16) are the equations of motion corresponding to $A_{\pm N}(k)$. (one could also obtain them by substitution of (3.8a), (3.9a) in (3.1)).

The commutation properties of the evolution operators will now be investigated. Recall equations (3.5):

$$\omega_{n,k}\omega^{-1} = \lambda^n\omega k\omega^{-1} - A_n(k) \qquad \forall n \in \mathbb{Z},\ k \in \underset{\sim}{k} \qquad (3.17)$$

(or alternatively use $\tilde{\omega}$, i.e. equations (3.7)) and consider the identity

$$([\partial_{n,k},\partial_{m,j}]\omega)\omega^{-1} = \partial_{n,k}(\omega_{m,j}\omega^{-1}) - \partial_{m,j}(\omega_{n,k}\omega^{-1})$$

$$+ [\omega_{m,j}\omega^{-1}, \omega_{n,k}\omega^{-1}] \qquad (3.18)$$

for all $m,n \in \mathbb{Z}$, $k,j \in \underset{\sim}{k}$. Using (3.17), one has

$$\partial_{n,k}(\omega_{m,j}\omega^{-1}) = \lambda^m[\lambda^n\omega k\omega^{-1} - A_n(k),\omega j\omega^{-1}] - \partial_{n,k}A_m(j) \quad (3.19)$$

One finds that (3.18) becomes

$$([\partial_{n,k},\partial_{m,j}]\omega)\omega^{-1} = \partial_{m,j}A_n(k) - \partial_{n,k}A_m(j)$$

$$- [A_n(k),A_m(j)] + \lambda^{n+m}\omega[k,j]\omega^{-1} \quad (3.20)$$

and so (using (3.17) to rewrite the last term in (3.20)) one can deduce that the relation

$$[\partial_{n,k},\partial_{m,j}] = \partial_{n+m,[k,j]} \quad (3.21)$$

is equivalent to the condition

$$\partial_{n,k}A_m(j) - \partial_{m,j}A_n(k) + [A_n(k),A_m(j)] = A_{n+m}([k,j]) \quad (3.22)$$

which must now be verified using (3.8a) and (3.9a). One needs to consider separately the cases where n,m are of the same or different sign. Suppose they are of different sign, i.e. $n = N$ (≥ 0) and $m = -M$ (≤ 0), and suppose $(N-M) \geq 0$. Split (3.22) into coefficients of λ^{-p} and λ^q (where $p,q \geq 0$):

$$\partial_{N,k}A^p_{-M}(j) + [A_N(k),A_{-M}(j)]_{-p} = 0 \quad (3.23a)$$

$$-\partial_{-M,j}A^q_N(k) + [A_N(k),A_{-M}(j)]_q = A^q_{N-M}([k,j]) \quad (3.23b)$$

Using (3.8a) and (3.9a), the left side of equation (3.23a) becomes

$$[\phi_{N,k}\phi^{-1},\phi(\Omega j\Omega^{-1})_{M-p}\phi^{-1}] + \phi([\Omega_{N,k}\Omega^{-1},\Omega j\Omega^{-1}])_{M-p}\phi^{-1}$$

$$+ \Sigma_{r=0}^{M-p}[(\omega k\omega^{-1})_{N-r},\phi(\Omega j\Omega^{-1})_{M-p-r}\phi^{-1}]$$

$$= \Sigma_{r=1}^{M-p}[(\omega k\omega^{-1})_{N-r},\phi(\Omega j\Omega^{-1})_{M-p-r}\phi^{-1}]$$

$$+ \Sigma_{r=1}^{M-p}\phi[(\Omega_{N,k}\Omega^{-1})_{r},(\Omega j\Omega^{-1})_{M-p-r}]\phi^{-1} \qquad \text{(using (3.10))}$$

$$= 0$$

as required (using (3.12b) and noting that $r < N$). The left side of (3.23b) becomes

$$-([\omega_{-M,j}\omega^{-1},\omega k\omega^{-1}])_{N-q} + \Sigma_{r=q+1}^{M+q}[(\omega k\omega^{-1})_{N-r},A_{-M}^{r-q}(j)]$$

$$= (\omega[k,j]\omega^{-1})_{N-M-q} \qquad \text{(using (3.9a), (3.12a))}$$

$$= A_{N-M}^{q}([k,j])$$

as required (using (3.8a)). The calculation for $(N-M)<0$ proceeds along similar lines, noting that for this case $A_{N-M}([k,j])$ has the form (3.9a). One can also check the cases where n,m in (3.22) are of the same sign. (The case $n,m \geq 0$ was considered in [1]). Having verified (3.22), one can conclude, from (3.21), that the evolution operators $\partial_{\pm N,k}$ provide a realization of the Kac-Moody algebra $\underset{\sim}{k} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$.

Notice that (3.22) is a generalization of the zero curvature condition (3.1). Putting (n,k) = (1,E) and recalling (3.14), equation (3.22) becomes

$$\partial_x A_m(j) - \partial_{m,j} A_x + [A_x, A_m(j)] = 0 \quad \forall m \in \mathbb{Z},\ j \in \underset{\sim}{k} \tag{3.24}$$

A less illuminating (but quicker) method of obtaining (3.21) to use ϕ in place of ω in (3.18). For example, using (3.10), (3.11) and equations (3.5), (3.7) one obtains

$$\begin{aligned} &([\partial_{N,k}, \partial_{-M,j}]\phi)\phi^{-1} \\ &= \Sigma_{r=1}^{N}[(\omega j \omega^{-1})_{r-M} - \phi(\Omega j \Omega^{-1})_{M-r}\phi^{-1}, (\omega k \omega^{-1})_{N-r}] \\ &+ \Sigma_{r=1}^{M}\phi[(\Omega k \Omega^{-1})_{r-N} - \phi^{-1}(\omega k \omega \quad)_{N-r}\phi, (\Omega j \Omega^{-1})_{M-r}]\phi^{-1} \end{aligned} \tag{3.25}$$

If (N-M)$\geq$0 then (3.25) becomes

$$\begin{aligned} ([\partial_{N,k}, \partial_{-M,j}]\phi)\phi^{-1} &= \Sigma_{s=0}^{N-M}[(\omega j \omega^{-1})_{N-M-s}, (\omega k \omega^{-1})_s] \\ &= -(\omega[k,j]\omega^{-1})_{N-M} \\ &= \phi_{N-M,[k,j]}\phi^{-1} \end{aligned} \tag{3.26}$$

with a similar result for (N-M)<0. One can also check the equal sign cases in the same fashion.

The evolution operators $\partial_{\pm N,k}$ can be extended in the obvious way, simply by replacing k by a general constant element $g \in \underset{\sim}{g}$, i.e.

$$A_n(g) = \lambda^n \omega g \omega^{-1} - \omega_{n,g}\omega^{-1} = \lambda^n \tilde{\omega} g \tilde{\omega}^{-1} - \tilde{\omega}_{n,g}\tilde{\omega}^{-1}$$

$$\forall\, n \in \mathbb{Z},\ g \in \underset{\sim}{g} \qquad (3.27)$$

One can then verify in the same way as earlier the equation

$$\partial_{n,g}A_m(h) - \partial_{m,h}A_n(g) + [A_n(g),A_m(h)] = A_{n+m}([g,h])$$

$$\forall\, n,m \in \mathbb{Z},\ g,h \in \underset{\sim}{g} \qquad (3.28)$$

which is equivalent to the condition

$$[\partial_{n,g},\partial_{m,h}] = \partial_{n+m,[g,h]} \qquad (3.29)$$

and so one obtains a realization of the Kac-Moody algebra $\underset{\sim}{g} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. Notice that for $m \in \underset{\sim}{m}$ the quantities $A_{\pm N}(m)$ do not satisfy the zero curvature condition, which is why they where not obtained in the original construction (where one sought solutions of (3.1)), and so the parameters ($\pm N$,m) cannot be regarded as true "times". Their interpretation will be discussed later.

4. HAMILTONIANS

The Hamiltonians $H_n(g)$ for the operators $\partial_{n,g}$ are deined by

$$\partial_{n,g}f = \{f,H_n(g)\} \qquad (4.1)$$

where f is any function, and the Poisson bracket is given by

$$\{f_1,f_2\} = \sum_\alpha \int \underset{-}{d}z(\partial f_1/\partial q^{-\alpha}(z).\partial f_2/\partial q^{\alpha}(z)$$

$$- \partial f_1/\partial q^{\alpha}(z).\partial f_2/\partial q^{-\alpha}(z)) \tag{4.2}$$

Hamilton's equations take the form

$$\partial_{n,g} q^{\alpha} = -\partial H_n(g)/\partial q^{-\alpha} \tag{4.3a}$$

$$\partial_{n,g} q^{-\alpha} = \partial H_n(g)/\partial q^{\alpha} \tag{4.3b}$$

Now, the Jacobi identity, together with (4.1) and (3.29), implies that (for any function f)

$$\{f\{H_n(g),H_m(h)\}\} = [\partial_{n,g},\partial_{m,h}]f$$

$$= \partial_{n+m,[g,h]}f$$

$$= \{f,H_{n+m}([g,h])\} \quad \forall n,m \in \mathbb{Z},\ g,h \in \underset{\sim}{g} \tag{4.4}$$

and so

$$\{H_n(g),H_m(h)\} = H_{n+m}([g,h]) + C^{g,h}_{n,m} \tag{4.5}$$

where $C^{g,h}_{n,m}$ is a constant. Since the Hamiltonians are only defined up to addition of constants, it is possible to arrange for the Poisson bracket to take the form [3]

$$\{H_n(g),H_m(h)\} = H_{n+m}([g,h]) + n\delta_{n,-m}\delta_{g,h}c \tag{4.6}$$

where c is a constant; i.e. the Hamiltonians provide a realization of the centrally extended algebra $g \otimes \mathbb{C}[\lambda,\lambda^{-1}] \oplus \mathbb{C}c$.

The Hamiltonian of the GNLS equation is $H_2(E)$, so equation (4.6) implies that $H_{\pm N}(k)$ are conserved quantities for the GNLS equation, except for $H_{-2}(E)$ if $c \neq 0$. Notice also that (4.6) and (4.1) imply

$$\partial_{\pm N,k}H_0(E) = 0 \qquad \forall N \geq 0,\ k \in \underset{\sim}{k} \tag{4.7}$$

and one can use (3.16a) (with N = 0, k = E) to deduce that

$$H_0(E) = -i\int q^{\alpha}q^{-\alpha} \tag{4.8}$$

(summation implied). Next, observe that

$$\int Tr(E[A_x^0,\partial_{n,g}A_x^0]) = i\int (q^{\alpha}_{n,g}q^{-\alpha} - q^{\alpha}q^{-\alpha}_{n,g})$$

$$= 2i\int q^{\alpha}{}_{n,g}q^{-\alpha} = -2i\int q^{-\alpha}_{n,g}q^{\alpha} \tag{4.9}$$

if $g \in \underset{\sim}{k}$ (using (4.7),(4.8)). Differentiation of (4.9) with respect to $q^{\pm\alpha}$ gives Hamilton's equations (4.3), so that one can use (3.16) to write

$$H_N(g) = -ia\int Tr(A_x^0\omega g\omega^{-1})_{N+1} \tag{4.10a}$$

$$H_{-N}(g) = ia\int Tr(A_x^0\psi(\Omega g\Omega^{-1})_{N-1}\psi^{-1}) \tag{4.10b}$$

where a = 1/2 if $g \in \underset{\sim}{k}$, and a = 1 if $g \in \underset{\sim}{m}$.

5. LINEARIZATION

For step operators $e_{\pm\alpha} \in \underset{\sim}{m}$, define the formal power series

$$\Gamma_{\pm\alpha}(\lambda) = \Sigma_{n=-\infty}^{\infty}\lambda^{-(n+1)}H_n(e_{\pm\alpha})$$

$$= i\int \mathrm{Tr}\left(A_x^0(\phi\Omega e_{\pm\alpha}\Omega^{-1}\phi^{-1} - \omega e_{\pm\alpha}\omega^{-1})\right) \tag{5.1}$$

(by (4.10)). Then (4.6) and (1.2) imply

$$\partial_{N,E}\Gamma_{\pm\alpha}(\lambda) = \{\Gamma_{\pm\alpha}(\lambda), H_N(E)\} = \pm i\lambda^N\Gamma_{\pm\alpha}(\lambda) \tag{5.2}$$

and so $\Gamma_{\pm\alpha}(\lambda)$ linearizes the equations of motion of the GNLS hierarchy.

Using the cyclic property of the trace, (5.1) can be written as

$$\Gamma_{\pm\alpha}(\lambda) = i\int\left((\Omega^{-1}\phi^{-1}A_x^0\phi\Omega)^{\mp\alpha} - (\omega^{-1}A_x^0\omega)^{\mp\alpha}\right) \tag{5.3}$$

where $(X)^{\mp\alpha}$ denotes the $e_{\mp\alpha}$ component of X. Now, the restriction to the compact or non-compact form corresponds to

$$(X)^{\alpha *} = \mp(X)^{-\alpha} \tag{5.4}$$

so that

$$\{\Gamma_\alpha(\lambda),\Gamma^*_\beta(\mu)\} = \mp\Sigma^\infty_{n=-\infty}\Sigma^\infty_{m=-\infty}\lambda^{-n-1}\mu^{-m-1}H_{n+m}([e_\alpha,e_{-\beta}]) \quad (5.5)$$

when $q^{-\alpha} = \mp q^{\alpha *}$. Also

$$\{\Gamma_\alpha(\lambda),\Gamma_\beta(\mu)\} = 0 \quad (5.6)$$

(since (1.2) implies that $[e_\alpha,e_\beta] = 0$). Equation (5.5) shows that the transformation

$$q^\alpha \to \Gamma_\alpha \quad (5.7)$$

is not canonical.

6. DISCUSSION

Recalling equation (3.14), one notes the following special cases of (3.29):

$$[\partial_x,\partial_{n,k}] = 0 \qquad \forall n \in \mathbb{Z},\ k \in \underset{\sim}{k} \quad (6.1a)$$

$$[\partial_x,\partial_{n,e_{\pm\alpha}}] = \mp i\partial_{1+n,e_{\pm\alpha}} \qquad \forall n \in \mathbb{Z},\ e_{\pm\alpha} \in \underset{\sim}{m} \quad (6.1b)$$

The parameters (n,k) could be regarded as "time" variables, but $(n,e_{\pm\alpha})$ cannot. Transformations which do not commute with translations are regarded, in the context of gauge theories, as "internal symmetries" [4]. The mutually commuting class

$\{\partial_{n,e_\alpha}\}$ can be thought of as the basis of an "internal space". It is the generator of translations in this space (Γ_α) which linearizes the GNLS system. It should be noted, however, that what one is really considering is the phase space of the system. The construction seems, in fact, to be a generalization of the conventional approach to the SU(2) non-linear Schrödinger equation [5], where one considers the so-called "monodromy matrix" whose diagonal elements give rise to conserved quantities, while the off-diagonal elements lead to the linearization of the system.

The use of the gauge transformation $\tilde{\omega}$ is similar to the method used by Olive and Turok for deriving the conserved quantities of the Toda equation [6]. In that case, a λ-independent local gauge transformation was composed with a local transformation of the form (1.25) so that the transformed gauge potential was a series belonging to $\underset{\sim}{k}$. In the present case, as was mentioned earlier, the λ-independent element ϕ is associated with the generalized Heisenberg ferromagnet (GHF) [2]. Consider the transformation

$$A_x \to \tilde{A}_x = \phi^{-1}A_x\phi + \phi^{-1}\phi_x = \lambda\phi^{-1}E\phi \tag{6.2a}$$

$$\begin{aligned} A_2(E) \to \tilde{A}_t &= \phi^{-1}A_2(E)\phi + \phi^{-1}\phi_{2,E} \\ &= \lambda^2\phi^{-1}E\phi + \lambda\phi^{-1}A_x^0\phi \end{aligned} \tag{6.2b}$$

(by (3.10)) and define

$$S = \phi^{-1}E\phi \tag{6.3}$$

Then

$$\partial_x S = \phi^{-1}[E,\phi_x\phi^{-1}]\phi = \phi^{-1}[A_x^0,E]\phi \tag{6.4}$$

and so

$$[S,S_x] = \phi^{-1}[E[A_x^0,E]]\phi = \phi^{-1}A_x^0\phi \tag{6.5}$$

i.e the transformed gauge potentials are

$$\tilde{A}_x = \lambda S \tag{6.6a}$$

$$\tilde{A}_t = \lambda^2 S + \lambda[S,S_x] \tag{6.6b}$$

and the zero curvature condition becomes the GHF equation

$$\partial_t S = [S,S_{xx}] \tag{6.7}$$

The conserved quantities which have been constructed for the GNLS system are non-local, because of the non-locality of the gauge transformations used to construct them. Non-local conserved quantities were constructed for the non-linear σ-model in [7], and it was shown in [8] that these are associated with infinitesimal transformations which form a centre-free Kac-Moody algebra. However, the charges themselves do not form an algebra [9], and the Kac-Moody symmetry is interpreted as a property of the solution space, rather than of the phase space. The infinitesimal symmetries of the SU(2) non-linear Schrödinger equation were investigated in [10]. In that construction, only the "positive" subalgebra was realized non-trivially.

The linearization of the GNLS system using step operators of a Kac-Moody algebra (equation (5.2)) seems to be related to the work of the Kyoto group [11], who use vertex operators to construct soliton solutions for a large class of equations. It would be interesting to establish the connection of these ideas with the approach of Adler and van Moerbeke [12]. Other topics worth pursuing include the investigation of the central term in (4.6) (e.g. the conditions under which it vanishes), and the quantization of the system. For the SU(2) case, the quantization of the action-angle variables (i.e. the canonical linearizing variables) gives the "Bethe ansatz" creation operators [13]. In the general case, quantization should lead to vertex operators of some sort.

The methods which have been presented here can be generalized to cover a wide range of integrable systems. This will be discussed in a subsequent paper.

REFERENCES

[1] Crumey, A.D.W.B.: Local and non-local conserved quantities for generalized non-linear Schrödinger equations. Commun. Math. Phys. 108, 631-646 (1987) (Chapter 2 of this thesis)

[2] Fordy, A.P., Kulish, P.P.: Nonlinear Schrödinger equations and simple Lie algebras. Commun. Math. Phys. 89, 427-443 (1983)

[3] Goddard, P., Olive, D.: Kac-Moody and Virasoro algebras in relation to quantum physics. DAMTP 86-5 (1986)

[4] Gervais, J.-L.: Gauge invariance over a group as the first principle of interacting string dynamics. LPTENS preprint 86/1 (1986)

[5] Faddeev, L.D. In: Recent advances in field theory and statistical mechanics, Les Houches, Session 39, Elsevier Sci. Publ.563-608 (1984)

[6] Olive, D., Turok, N.: Local conserved densities and zero curvature conditions for Toda lattice field theories. Nucl. Phys. B257[FS14], 277 (1985)

[7] Luscher, M., Pohlmeyer, K.: Scattering of massless lumps and non-local charges in the two-dimensional classical non-linear σ-model. Nucl. Phys. B137, 46-54 (1978)

[8] Dolan, L: Kac-Moody algebra is hidden symmetry of chiral models. Phys. Rev. Lett. 47, 1371-1374 (1981)

[9] de Vega, H.J., Eichenherr, H., Maillet, J.M.: Classical and quantum algebras of non-local charges in σ-models. Commun. Math. Phys. 92 507-524 (1984)

[10] Eichenherr, H.: Symmetry algebras of the Heisenberg model and the non-linear Schrödinger equation. Phys. Lett. 115B, 385-388 (1982)

[11] For a review, see: Jimbo, M., Miwa, T.: Solitons and infinite dimensional Lie algebras. Publ. RIMS, Kyoto Univ. 19, 943-1001 (1983)

[12] Adler, M., van Moerbeke, P.: Linearization of Hamiltonian systems, Jacobi varieties and representation theory. Adv. in Math. 38, 318-379 (1980)

[13] For a review, see: Thacker, H.B.: Exact integrability in quantum field theory and statistical systems. Rev. Mod. Phys. 53 no.2, 253-285 (1981)

CHAPTER 4: INTEGRABLE DYNAMICAL SYSTEMS AND KAC-MOODY ALGEBRAS

1. INTRODUCTION

Ever since the work of Zakharov and Shabat [1], it has been clear that a wide range of physically important non-linear evolution equations in 1+1 dimensions may be written as the compatibility condition of a pair of linear equations, i.e. as a zero curvature equation

$$[\partial_x + A_x, \partial_t + A_t] = 0 \tag{1.1}$$

where the "gauge potentials" A_x, A_t are matrices, usually written as polynomials (or rational functions) in a complex parameter λ (which does not appear in the equation of motion) with coefficients in a Lie algebra $\underset{\sim}{g}$.

For a given equation of motion, it may be very difficult (and will usually be impossible) to obtain a zero-curvature representation. A more reasonable approach is to try and classify the sort of equations which can arise from (1.1). If A_x is fixed, then one may seek A_t to arbitrary order by equating powers of λ to zero in (1.1). This will give rise to a hierarchy of equations of motion. The problem of finding A_t is made easier by the invariance of (1.1) under a "gauge transformation"

$$A_\mu \to \gamma^{-1} A_\mu \gamma + \gamma^{-1} \gamma_\mu \equiv a_\mu \tag{1.2}$$

(where $\gamma_\mu \equiv \partial_\mu \gamma$). The matrix coefficients of γ belong to the Lie group G of $\underset{\sim}{g}$. It may be possible to construct a transformation $A_x \to a_x$ which allows the zero curvature condition (1.1) to be solved for a_t. Then A_t is obtained via the inverse transformation.

In [2,3], this was carried out for the generalized non-linear Schrödinger hierarchy [4], in which the gauge potentials are polynomials with coefficients in a symmetric Lie algebra $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$. By transforming A_x to a constant element, it was possible to obtain a closed expression for A_t to any order in powers of λ, and so the commutation properties of the hierarchy of evolution operators (∂_t) could be investigated. They were found to provide a realization of the centre free Kac-Moody algebra $\underset{\sim}{k} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. It was then possible to construct a further hierarchy of equations so as to obtain the full algebra $\underset{\sim}{g} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. The evolution operators corresponding to the subspace $\underset{\sim}{m}$ were found not to commute with the translation operator ∂_x, and were regarded as translations in "internal" dimensions, in analogy with the situation in gauge field theories [5]. The Hamiltonians of the evolution operators were found to provide a classical "current algebra" realization of the same algebra. (The central term which arose will, in this paper, be seen to vanish). The Hamiltonians corresponding to the subalgebra $\underset{\sim}{k}$ could then be regarded as conserved quantities for the generalized non-linear Schrödinger equation, while those corresponding to $\underset{\sim}{m}$ could be used to linearize the equation of motion. The system was thus shown to be completely integrable.

In the present work, constant gauge potentials will be used to obtain other polynomial pairs (A_x,A_t) associated with integrable dynamical systems. One begins with a non-commutative version of the zero curvature condition

$$\partial_{n_1,g_1} A_{n_2,g_2} - \partial_{n_2,g_2} A_{n_1,g_1} + [A_{n_1,g_1}, A_{n_2,g_2}] = A_{n_1+n_2,[g_1,g_2]} \tag{1.3}$$

where $\partial_{n,g}$ ($n \in \mathbb{Z}$, $g \in \underset{\sim}{g}$) form a centre free Kac-Moody algebra, and are regarded as operators of differentiation with respect to parameters $t_{n,g}$. One of these parameters (with g an element of a Cartan subalgebra $\underset{\sim}{h}$ of $\underset{\sim}{g}$) will be chosen as "space", labelled x. Those operators $\partial_{n,k}$ which commute with ∂_x may be regarded as defining a hierarchy of evolution equations, while those which do not are regarded as "internal" translations. The Hamiltonians will be shown to provide a realization of the same Kac-Moody algebra, and the dynamical systems defined by $\partial_{n,h}$ ($h \in \underset{\sim}{h}$) will be completely integrable. The equations of motion are, in general, non-local. The construction of such systems will be described in Section 2. The key ingredient is the gauge transformation which takes constant solutions of (1.3) to dynamical ones. This is obtained from a "gauge equation", which is discussed in detail in Section 3. In particular, it is shown that for any choice of x, it is always possible to construct a hierarchy of integrable systems.
Also, if A_x is expressed locally in terms of dynamical fields and their derivatives at a point, then there will always be a hierarchy of local equations of motion (and hence there will be local conserved quantities). In Sections 4 and 5, some examples will be given of equations of motion which arise from the construction.

2. METHOD

Let $\underset{\sim}{g}$ be a semisimple Lie algebra. The "loop algebra" $\underset{\sim}{g}'$ is the centre-free Kac-Moody algebra $\underset{\sim}{g} \otimes \mathbb{C}[\lambda,\lambda^{-1}]$, where $\mathbb{C}[\lambda,\lambda^{-1}]$ is the algebra of Laurent polynomials in the complex parameter λ. $\underset{\sim}{g}'$ may be represented as the class of elements $\partial_{n,g}$ ($n \in \mathbb{Z}$, $g \in \underset{\sim}{g}$), equipped with the bracket

$$[\partial_{n_1,g_1}, \partial_{n_2,g_2}] = \partial_{n_1+n_2,[g_1,g_2]} \tag{2.1}$$

Now one wishes to construct a realization of $\underset{\sim}{g}'$ as a family of differential operators acting on a space F of functions (dynamical fields). The elements $\partial_{n,g}$ will be regarded as defining differentiation (evolution) with respect to the parameters $t_{n,g}$. Such a realization will be called "dynamical".

If F is equipped with a Poisson bracket $\{\ ,\ \}$ then one may define the Hamiltonians $H_{n,g}$ F by

$$\partial_{n,g}u = \{u, H_{n,g}\} \qquad \forall u \in F \tag{2.2}$$

Then the Jacobi identity implies that

$$\begin{aligned}\{u,\{H_{n_1,g_2}, H_{n_2,g_2}\}\} &= [\partial_{n_1,g_1}, \partial_{n_2,g_2}]u \\ &= \partial_{n_1+n_2,[g_1+g_2]}u \\ &= \{u, H_{n_1+n_2,[g_1,g_2]}\} \qquad \forall u \in F\end{aligned} \tag{2.3}$$

and so

$$\{H_{n_1,g_1}, H_{n_2,g_2}\} = H_{n_1+n_2,[g_1,g_2]} + n_1\delta_{n_1,-n_2}\delta_{g_1,g_2}c \qquad (2.4)$$

where $c \in \mathbb{C}$ is a constant. (The constant term has been written in its most general form [6], which can always be arrived at by a suitable redefinition of the Hamiltonians, since they are only defined up to addition of constants.) This means that for any dynamical realization there is an associated "Hamiltonian realization" of the algebra (with central extension, if $c \neq 0$).

For some $p \in \mathbb{Z}$, $E \in \underset{\sim}{g}$, let $t_{p,E}$ be a distinguished parameter which will be called the "space dimension", labelled x. Without loss of generality, it will be assumed that p is positive and E belongs to a Cartan subalgebra $\underset{\sim}{h}$ of $\underset{\sim}{g}$. Then $\underset{\sim}{g}$ may be decomposed as

$$\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m} \qquad (2.5)$$

where $\underset{\sim}{k}$ is the centralizer of E (see Appendix). The collection

$$\underline{T} = \{t_{n,k}: n \in \mathbb{Z},\ k \in \underset{\sim}{k}\} \qquad (2.6)$$

will be called the "evolution space". Any element of $\underline{T}$ may be singled out and regarded as the "time dimension", labelled t. Then the manifold spanned by (x,t) will be called the "physical space", and the operator ∂_t will define the equation of motion for any function u(x,t) F. The whole class $\partial_{n,k}$ (where $t_{n,k} \in \underline{T}$) defines a collection of equations of motion, which will be referred to as the "dynamical equations". The equations defined by the operators $\partial_{n,E}$ ($n \in \mathbb{Z}$) will be called the "central hierarchy", while those defined by $\partial_{N,E}$ ($N \geq 0$) will be called the "fundamental hierarchy".

The collection

$$\underline{I} = \{t_{n,m}: n \in \mathbb{Z},\ m \in \underset{\sim}{m}\} \tag{2.7}$$

consists of parameters whose evolution operators $\partial_{n,m}$ do not commute with the spatial translation operator ∂_x (i.e. momentum is not conserved for the evolution equations). $\underline{I}$ will be called the "internal" space, and the operators $\partial_{n,m}$ will be regarded as defining translations in the "internal" (non-physical) dimensions $t_{n,m}$. For all the dynamical equations, momentum ($H_{p,E}$) will be conserved. In particular, this means that

$$0 = \partial_{-p,E}H_{p,E} = \{H_{p,E}, H_{-p,E}\} = pc \tag{2.8}$$

(from (2.4)), and so the central term actually vanishes (if $p \neq 0$).

Next notice that (2.4) implies

$$\{H_{n,k}\ ,\ H_{m,E}\} = 0 \qquad \forall n,m \in \mathbb{Z},\ k \in \underset{\sim}{k} \tag{2.9}$$

i.e. the Hamiltonians $H_{n,k}$ are conserved quantities for the equations of the central hierarchy. The subset consisting of $H_{n,h}$ ($n \in \mathbb{Z}$, $h \in \underset{\sim}{h}$) forms a maximal involutive class of conserved quantities. Furthermore, for any step operator

$e_\alpha \in \underset{\sim}{m}$, one may define

$$\Gamma_\alpha \equiv \Sigma_{n=-\infty}^{\infty} \lambda^{-n} H_{n,e_\alpha} \tag{2.10}$$

(where λ is a formal parameter). Then, using (2.4) (and the notation $[E,e_\alpha] = (\alpha.E)e_\alpha$),

$$\begin{aligned} \partial_{n,E}\Gamma_\alpha &= \Sigma_{m=-\infty}^{\infty}\lambda^{-m}\{H_{m,e_\alpha}, H_{n,E}\} \\ &= -\Sigma_{m=-\infty}^{\infty}\lambda^{-m}(\alpha.E)H_{m+n,e_\alpha} \\ &= -\lambda^{n}(\alpha.E)\Gamma_\alpha \qquad \forall n \in \mathbb{Z},\ e_\alpha \in \underset{\sim}{m} \end{aligned} \tag{2.11}$$

and so the equations of motion are linearized. Eqs.(2.9),(2.11) show that the equations of the central hierarchy are completely integrable. For the equations defined by $\partial_{n,h}$ ($n \in \mathbb{Z}$, $h \in \underset{\sim}{h}$, $h \neq E$), there may be fewer conserved quantities (i.e. only those of the maximal involutive class), but the linearization (2.11) still applies. For the equations defined by $\partial_{n,e}$ ($n \in \mathbb{Z}$, $e \in \underset{\sim}{k}, e \notin \underset{\sim}{h}$), there are still infinitely many conserved quantities ($H_{m,e}$: $m \in \mathbb{Z}$), but the equations are not linearized by (2.10).

In order to construct a dynamical realization, one begins by considering the equation

$$\partial_{n_1,g_1}A_{n_2,g_2} - \partial_{n_2,g_2}A_{n_1,g_1} + [A_{n_1,g_1}, A_{n_2,g_2}] \\ = A_{n_1+n_2,[g_1,g_2]} \tag{2.12}$$

for an unknown class of "potentials" $A_{n,g} \in F \otimes g \otimes \mathbb{C}[\lambda,\lambda^{-1}]$. (Notice that when $[g_1,g_2] = 0$, Eq.(2.12) becomes a zero curvature condition).

One seeks solutions $A_{n,g}$ for (2.12). The simplest example is the constant solution

$$A_{n,g} = \lambda^n g \tag{2.13}$$

Now consider the "gauge transformation"

$$A_{n,g} \to \gamma A_{n,g}\gamma^{-1} - \gamma_{n,g}\gamma^{-1} \equiv A^{(\gamma)}_{n,g} \tag{2.14}$$

where γ belongs to the loop group, with coefficients in F (and $\gamma_{n,g} \equiv \partial_{n,g}\gamma$). Making use of the identities

$$\partial_{n,g}(\gamma^{-1}) = -\gamma^{-1}\gamma_{n,g}\gamma^{-1} \tag{2.15a}$$

$$\partial_{n,g}(\gamma A\gamma^{-1}) = [\gamma_{n,g}\gamma^{-1},\ \gamma A\gamma^{-1}] + \gamma\partial_{n,g}A\gamma^{-1} \quad \forall A \in g \tag{2.15b}$$

Eq.(2.12) is transformed to

$$\partial_{n_1,g_1}A^{(\gamma)}_{n_2,g_2} - \partial_{n_2,g_2}A^{(\gamma)}_{n_1,g_1} + [A^{(\gamma)}_{n_1,g_1},\ A^{(\gamma)}_{n_2,g_2}]$$

$$= A^{(\gamma)}_{n_1+n_2,[g_1,g_2]} \tag{2.16}$$

i.e. the equation (2.12) is form-invariant under (2.14). Therefore, using (2.13) in (2.14), a solution is

$$A_{n,g} = \lambda^n\gamma g\gamma^{-1} - \gamma_{n,g}\gamma^{-1} \tag{2.17}$$

where γ is arbitrary. Now choose two possible forms:

$$\gamma_1 = \phi\omega = \phi\exp\Sigma_{n=1}^{\infty}\lambda^{-n}\omega_n \tag{2.18a}$$

$$\gamma_2 = \Psi\Omega = \Psi\exp\Sigma_{n=1}^{\infty}\lambda^{n}\Omega_n \tag{2.18b}$$

where ϕ,Ψ are independent of λ and $\omega_n,\Omega_n \in F \otimes \underset{\sim}{g}$. With these two choices, $A_{n,g}$ will be an infinite series in either descending or ascending powers.

Now impose the condition that $A_{n,g}$ is the same for either choice of γ. Then $A_{n,g}$ must be a polynomial of degree n in only positive (negative) powers of λ if n is positive (negative). Let N be a positive integer. Then (using (2.15a))

$$A_{N,g} = \lambda^N\phi\omega g\omega^{-1}\phi^{-1} - \phi\omega_{N,g}\omega^{-1}\phi^{-1} - \phi_{N,g}\phi^{-1} \tag{2.19a}$$

$$= \lambda^N\Psi\Omega g\Omega^{-1}\Psi^{-1} - \Psi\Omega_{N,g}\Omega^{-1}\Psi^{-1} - \Psi_{N,g}\Psi^{-1} \tag{2.19b}$$

$$\equiv \Sigma_{n=0}^{N}\lambda^n A_{N,g}^n \tag{2.19c}$$

Equating coefficients of powers of λ, one obtains

$$(\omega_{N,g}\omega^{-1})_n = (\omega g\omega^{-1})_{N+n} \qquad \forall n>0 \tag{2.20a}$$

$$A_{N,g} = \Sigma_{n=0}^{N}\lambda^n\phi(\omega g\omega^{-1})_{N-n}\phi^{-1} - \phi_{N,g}\phi^{-1} \tag{2.20b}$$

$$(\Omega_{N,g}\Omega^{-1})_n = (\Omega g\Omega^{-1})_{n-N} - \Psi^{-1}A_{N,g}^n\Psi \qquad \forall n>0 \tag{2.20c}$$

$$\Psi_{N,g}\Psi^{-1} = \phi_{N,g}\phi^{-1} - \phi(\omega g\omega^{-1})_N\phi^{-1} \qquad \forall N>0 \tag{2.20d}$$

$$\Psi_{N,g}\Psi^{-1} = \phi_{N,g}\phi^{-1} - \phi(\omega g\omega^{-1})_N\phi^{-1} \qquad \forall N>0 \qquad (2.20d)$$

$$\Psi_{0,g}\Psi^{-1} = \phi_{0,g}\phi^{-1} - \phi g\phi^{-1} + \Psi g\Psi^{-1} \qquad (2.20e)$$

where $(f(\omega))_n$ denotes the coefficient of λ^{-n} (=0 if n<0) and $(f(\Omega))_n$ denotes the coefficient of λ^n (=0 if n<0). Repeating the process with -N in (2.19), one obtains

$$(\Omega_{-N,g}\Omega^{-1})_n = (\Omega g\Omega^{-1})_{N+n} \qquad \forall n>0 \qquad (2.21a)$$

$$A_{-N,g} = \Sigma_{n=0}^{N}\lambda^{-n}\Psi(\Omega g\Omega^{-1})_{N-n}\Psi^{-1} - \Psi_{-N,g}\Psi^{-1} \qquad (2.21b)$$

$$(\omega_{-N,g}\omega^{-1})_n = (\omega g\omega^{-1})_{n-N} - \phi^{-1}A^n_{-N,g}\phi \qquad \forall n>0 \qquad (2.21c)$$

$$\Psi_{-N,g}\Psi^{-1} = \phi_{-N,g}\phi^{-1} + \Psi(\Omega g\Omega^{-1})_N\Psi^{-1} \qquad \forall N>0 \qquad (2.21d)$$

where $A^n_{-N,g}$ denotes the coefficient of λ^{-n} (=0 if n>N).

Now write $x \equiv t_{p,E}$. From Eq.(2.20), one has

$$(\omega_x\omega^{-1})_n = (\omega E\omega^{-1})_{p+n} \qquad \forall n>0 \qquad (2.22a)$$

$$A_x = \Sigma_{n=0}^{p}\lambda^n\phi(\omega E\omega^{-1})_{p-n}\phi^{-1} - \phi_x\phi^{-1} \qquad (2.22b)$$

$$(\Omega_x\Omega^{-1})_n = (\Omega E\Omega^{-1})_{n-p} - \Psi^{-1}A^n_x\Psi \qquad \forall n>0 \qquad (2.22c)$$

Eq.(2.22a) defines the generators ω_n as functions of x. The coefficients $\omega_1,\ldots,\omega_{p-1};\omega_p^{\underset{\sim}{m}}$ (where the superscript is used to denote the $\underset{\sim}{m}$ component), which appear in A_x, are not uniquely determined by (2.22a), and will be called the "dynamical sector". It will be seen that the dynamical sector may be chosen so that ω_n is determined to all orders by (2.22a). ϕ is undetermined, and may be chosen arbitrarily. Ψ, also, is not explicitly determined, but is fixed by (2.20d), (2.20e), (2.21d). The generators Ω_n are determined by (2.22c) in terms of Ψ and the dynamical sector.

The evolution of ϕ is subject to the consistency condition

$$\partial_{n_1,g_1}(\phi^{-1}\phi_{n_2,g_2}) - \partial_{n_1,g_1}(\phi^{-1}\phi_{n_2,g_2})$$

$$+ [\phi^{-1}\phi_{n_1,g_1},\ \phi^{-1}\phi_{n_2,g_2}] = \phi^{-1}\phi_{n_1+n_2,[g_1,g_2]} \qquad (2.23)$$

i.e. $\phi^{-1}\phi_{n,g}$ is a λ-independent solution of Eq.(2.12). If ϕ is chosen to be the identity, then (from (2.20d,e),(2.21d)) Ψ is fixed by

$$\Psi_{N,g}\Psi^{-1} = -(\omega g\omega^{-1})_N \qquad \forall N>0 \qquad (2.24a)$$

$$\Psi_{0,g}\Psi^{-1} = \Psi g\Psi^{-1} \qquad (2.24b)$$

$$\Psi_{-N,g}\Psi^{-1} = \Psi(\Omega g\Omega^{-1})_N\Psi^{-1} \qquad \forall N>0 \qquad (2.24c)$$

This case will be called the "basic gauge". Now, since $\phi^{-1}\phi_{n,g}$ is a solution of Eq.(2.12), one can choose

$$\phi^{-1}\phi_{n,g} = A^{(\phi)0}_{n,g} \qquad \forall n\in\mathbb{Z},\ g\in\mathfrak{g} \qquad (2.25)$$

where $A^{(\phi)}_{n,g}$ is a solution in any gauge ϕ; i.e. (from (2.20b), (2.21b,d) with ψ replaced by ϕ)

$$\psi^{-1}\psi_{N,g} = \phi(\omega g\omega^{-1})_N\phi^{-1} - \phi_{N,g}\phi^{-1} \tag{2.26a}$$

$$\psi^{-1}\psi_{-N,g} = -\phi_{-N,g}\phi^{-1} \qquad \forall N>0 \tag{2.26b}$$

Letting ϕ be the basic gauge (i.e. ϕ=1), one obtains the "principal gauge":

$$\psi^{-1}\psi_{N,g} = (\omega g\omega^{-1})_N \qquad \forall N\geq 0 \tag{2.27a}$$

$$\psi^{-1}\psi_{-N,g} = 0 \qquad \forall N>0 \tag{2.27b}$$

In this gauge, Eqs.(2.20d,e),(2.21d) become

$$\Psi^{-1}\Psi_{N,g} = 0 \qquad \forall N>0 \tag{2.28a}$$

$$\Psi^{-1}\Psi_{-N,g} = (\Omega g\Omega^{-1})_N \qquad \forall N\geq 0 \tag{2.28b}$$

(Letting ϕ be the principal gauge in (2.26), one returns to the basic gauge).

One may also fix ψ by choosing $\psi_x\psi^{-1}$ as a function of the dynamical sector. Then the dynamical equations for ψ may be written as

$$\psi^{-1}\psi_{n,k} = \partial^{-1}(\psi^{-1}\partial_{n,k}(\psi_x\psi^{-1})\psi) \tag{2.29}$$

which satisfies Eq.(2.23) for $g \in \underset{\sim}{k}$. (In general, one cannot extend the non-local definition (2.29) to the internal translations $\phi^{-1}\phi_{n,m}$ ($m \in \underset{\sim}{m}$). Eq.(2.23) may no longer hold, since ∂_x does not commute with these translations). Gauges for which $\phi^{-1}\phi_{N,E}$ (the fundamental hierarchy) is local will be called "local gauges" (although ϕ itself is a non-local gauge transformation).

3. THE GAUGE EQUATIONS

It will now be shown how ω and Ω are determined from the "gauge equations" (2.22a),(2.22c)

$$(\omega_x\omega^{-1})_n = (\omega E\omega^{-1})_{p+n} \tag{3.1}$$

$$(\Omega_x\Omega^{-1})_n = (\Omega E\Omega^{-1})_{n-p} - \Psi^{-1}\phi(\omega E\omega^{-1})_{p-n}\phi^{-1}\Psi \tag{3.2}$$

for all n>0. Recalling the definitions (2.18), one can expand in powers of λ to obtain the identities

$$(\omega_\mu\omega^{-1})_n = \Sigma_{r=1}^{n}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\ldots[\omega_{k_{r-1}},\ \partial_\mu\omega_{k_r}]\ldots]] \tag{3.3a}$$

$$(\omega g\omega^{-1})_n = \Sigma_{r=1}^{n}(r!)^{-1}\Sigma_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\ldots[\omega_{k_r},\ g]\ldots]] \quad \forall g \in \underset{\sim}{g} \tag{3.3b}$$

(and similarly for Ω). If p=0, then (3.1),(3.2) become (using (3.3))

$$\partial_x \omega_n = [\omega_n, E] \tag{3.4a}$$

$$\partial_x \Omega_n = [\Omega_n, E] \tag{3.4b}$$

i.e. one can choose $\omega_n = \Omega_n = 0$ for all n. Henceforth, it will be assumed that $p \geq 1$.

Putting n=1 in (3.2), one obtains (using (3.3) again)

$$\partial_x \Omega_1 = -\Psi^{-1}\phi(\omega E \omega^{-1})_{p-1}\phi^{-1}\Psi \qquad \forall\, p>1 \tag{3.5a}$$

$$\partial_x \Omega_1 = E - \Psi^{-1}\phi E \phi^{-1}\Psi \qquad p=1 \tag{3.5b}$$

and so Ω_1 is determined (non-locally) in terms of ω, ϕ, Ψ. Now suppose that all $\Omega_{j<n}$ are determined. Then (3.2) gives

$$\partial_x \Omega_n + \Sigma_1(\text{terms in } \Omega_{j<n})$$

$$= \Sigma_2(\text{terms in } \Omega_{j<n}) - \Psi^{-1}\phi(\omega E \omega^{-1})_{p-n}\phi^{-1}\Psi \tag{3.6}$$

and so Ω_n is determined to all orders in terms of ω, ϕ, Ψ.

For the gauge equation (3.1), the situation is more complicated. First, let p=1. Then, putting n=1 in Eq.(3.1)

$$\partial_x \omega_1 = [\omega_2, E] + 1/2[\omega_1[\omega_1, E]] \tag{3.7}$$

This may be split into $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components (using (A.5)):

$$\partial_x \omega_1^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}} \tag{3.8a}$$

$$[\omega_2, E] = \partial_x \omega_1^{\underset{\sim}{m}} - 1/2[\omega_1^{\underset{\sim}{k}}[\omega_1, E]] - 1/2[\omega_1^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{m}} \qquad (3.8b)$$

Eq.(3.8a) determines $\omega_1^{\underset{\sim}{k}}$ (non-locally) in terms of the dynamical sector $\omega_1^{\underset{\sim}{m}}$. Then $\omega_2^{\underset{\sim}{m}}$ is determined by Eq.(3.8b). Now suppose that $\omega_1, \ldots, \omega_{n-1}; \omega_n^{\underset{\sim}{m}}$ are determined. Then, using (3.3), Eq.(3.1) (with p=1) becomes

$$\partial_x \omega_n + \Sigma_1 = [\omega_{n+1}, E] + 1/2[\omega_n[\omega_1, E]] + 1/2[\omega_1[\omega_n, E]] + \Sigma_2 \qquad (3.9)$$

(where Σ_j are sums of commutators involving $\omega_{m<n}$). Then the $\underset{\sim}{k}$ and $\underset{\sim}{m}$ components are

$$\partial_x \omega_n^{\underset{\sim}{k}} = 1/2[\omega_n^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}} + 1/2[\omega_1^{\underset{\sim}{m}}[\omega_n, E]]^{\underset{\sim}{k}} + \Sigma_3 \qquad (3.10a)$$

$$[\omega_{n+1}, E] = \partial_x \omega_n^{\underset{\sim}{m}} - 1/2[\omega_n[\omega_1, E]]^{\underset{\sim}{m}} - 1/2[\omega_1[\omega_n, E]]^{\underset{\sim}{m}} + \Sigma_4 \qquad (3.10b)$$

Eq.(3.10a) determines $\omega_n^{\underset{\sim}{k}}$ non-locally. Then $\omega_{n+1}^{\underset{\sim}{m}}$ is determined by Eq.(3.10b). So ω_n is determined to all orders.

Now consider the case p=2. Putting n=1 in Eq.(3.1), the $\underset{\sim}{k}$ component is

$$\partial_x \omega_1^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_2, E]]^{\underset{\sim}{k}} + 1/2[\omega_2^{\underset{\sim}{m}}[\omega_1, E]]^{\underset{\sim}{k}} + 1/6[\omega_1[\omega_1[\omega_1, E]]]^{\underset{\sim}{k}} \qquad (3.11)$$

One is free to choose the dynamical sector $(\omega_1, \omega_2^{\underset{\sim}{m}})$, subject to Eq.(3.11). (One such choice will be discussed later on). Now suppose that $\omega_{n<N};\omega_N^{\underset{\sim}{m}}$ are determined in terms of the dynamical sector. Then the $\underset{\sim}{m}$ component of Eq.(3.1) with p=2, n=N-1 takes the form

$$[\omega_{N+1},E] + 1/2[\omega_N^{\underset{\sim}{k}}[\omega_1,E]] = \Sigma_1 \tag{3.12}$$

(where Σ_j involves the coefficients $\omega_{n<N};\omega_N^{\underset{\sim}{m}}$). Now put n=N in Eq.(3.1) (with p=2). The $\underset{\sim}{k}$ component gives

$$\partial_x\omega_N^{\underset{\sim}{k}} = 1/2[\omega_1^{\underset{\sim}{m}}[\omega_{N+1},E]]^{\underset{\sim}{k}} + 1/2[\omega_{N+1}^{\underset{\sim}{m}}[\omega_1,E]]^{\underset{\sim}{k}}$$

$$+ 1/6[\omega_N^{\underset{\sim}{k}}[\omega_1^{\underset{\sim}{m}}[\omega_1,E]]]^{\underset{\sim}{k}} + 1/6[\omega_1^{\underset{\sim}{m}}[\omega_N^{\underset{\sim}{k}}[\omega_1,E]]]^{\underset{\sim}{k}} + \Sigma_2 \tag{3.13}$$

Substituting (3.12) and using (A.8), this can be rewritten

$$\partial_x\omega_N^{\underset{\sim}{k}} = -1/3[\omega_1^{\underset{\sim}{m}}[\omega_N^{\underset{\sim}{k}}[\omega_1,E]]]^{\underset{\sim}{k}} + 1/6[\omega_N^{\underset{\sim}{k}}[\omega_1^{\underset{\sim}{m}}[\omega_1,E]]]^{\underset{\sim}{k}} + \Sigma_3 \tag{3.14}$$

$$= \Sigma_3 \tag{3.15}$$

using (A.10). This determines $\omega_N^{\underset{\sim}{k}}$, and then $\omega_{N+1}^{\underset{\sim}{m}}$ is determined by (3.12). So ω_n is determined to all orders.

For p>2, the sort of cancellation which appears in (3.14) will not occur, and one must choose the dynamical sector so that ω_n may be determined to all orders in terms of $\omega_{j<n}$. Such a choice will not be unique. For example, for any p>2 let q be a positive integer such that $2q\leq p-1$, and write

$$q = 2r + \sigma \tag{3.16}$$

where σ $\{0,1\}$. Now choose

$$\omega_1,\ldots\omega_{p-q-1} = 0 \qquad (3.17a)$$

$$\omega^{\underset{\sim}{m}}_{p-q},\ldots,\omega^{\underset{\sim}{m}}_{p-r-1} = 0 \qquad (3.17b)$$

One must check the consistency of (3.17). For $1\leq n\leq p$ the gauge equation (3.1) becomes (using (3.3))

$$\partial_x\omega_n = [\omega_{n+p},E] + 1/2\Sigma^{n+q}_{j=p-r}[\omega_{n+p-j}[\omega_j,E]] \qquad (3.18)$$

For $1\leq n\leq p-q-r-1$ this reduces to

$$[\omega_{n+p},E] = 0 \qquad (3.19)$$

i.e. $\omega^{\underset{\sim}{m}}_{p+1},\ldots,\omega^{\underset{\sim}{m}}_{2p-q-r-1} = 0$. Then, for $p-q-r\leq n\leq p-q-1$, Eq.(3.18) gives

$$[\omega_{n+p},E] = -1/2\Sigma^{n+q}_{j=p-r}[\omega_{n+p-j}[\omega_j,E]] \qquad (3.20)$$

The $\underset{\sim}{k}$ component of this equation is zero, by (3.17b). So $\omega^{\underset{\sim}{m}}_{2p-q-r},\ldots,\omega^{\underset{\sim}{m}}_{2p-q-1}$ are now known in terms of $\omega_{p-q},\ldots,\omega_{p-1}$.

For $p-q\leq n\leq p-r-1$, the $\underset{\sim}{k}$ component of Eq.(3.18) is

$$\partial_x\omega_n = 1/2\Sigma^{n+r}_{j=p-r}[\omega^{\underset{\sim}{m}}_{n+p-j}[\omega_j,E]]^{\underset{\sim}{k}} \qquad (3.21)$$

(using (3.17)). This determines $\omega_{p-q},\ldots,\omega_{p-r-1}$ in terms of $\omega^{\underset{\sim}{m}}_{p-r},\ldots,\omega^{\underset{\sim}{m}}_{p-1}$. The $\underset{\sim}{m}$ component is

$$[\omega_{n+p},E] = -1/2\Sigma_{j=p-r}^{n+q}[\omega^{\tilde{k}}_{n+p-j}[\omega_j,E]]$$

$$- 1/2\Sigma_{j=p-r}^{n+r}[\omega^{\tilde{m}}_{n+p-j}[\omega_j,E]]^{\tilde{m}} \qquad (3.22)$$

For $p-r\leq n\leq p$ the $\tilde{k}$ component of (3.18) becomes (using (3.17),(3.19))

$$\partial_x\omega^{\tilde{k}}_n = 1/2\Sigma_{j=p-r}^{n}[\omega^{\tilde{m}}_{n+p-j}[\omega_j,E]]^{\tilde{k}} \qquad (3.23)$$

and so $\omega^{\tilde{k}}_{p-r},\ldots,\omega^{\tilde{k}}_p$ are determined. Then $\omega^{\tilde{m}}_{2p-q},\ldots,\omega^{\tilde{m}}_{2p-r-1}$ are determined by (3.22).

So far, $\omega_1,\ldots,\omega_p;\omega^{\tilde{m}}_{p+1},\ldots,\omega^{\tilde{m}}_{2p-r-1}$ have been consistently determined in terms of $\omega^{\tilde{m}}_{p-r},\ldots,\omega^{\tilde{m}}_p$ (which may be regarded as the dynamical fields of the system). Now suppose that $\omega_{j<N};\omega^{\tilde{m}}_N,\ldots,\omega^{\tilde{m}}_{N+p-r-2}$ have been determined (where $N\geq p+1$). The $\tilde{k}$ component of the gauge equation (3.1) (with n=N) may be written as

$$\partial_x\omega^{\tilde{k}}_N = 1/2\Sigma_{j=p-q}^{N+q}[\omega^{\tilde{m}}_j[\omega_{N+p-j},E]]^{\tilde{k}} + \Sigma(\text{terms in } \omega_{j<N}) \qquad (3.24)$$

and so $\omega^{\tilde{k}}_N$ is determined. The $\tilde{m}$ component of (3.1) with n=N-r-1 has the form

$$[\omega_{N+p-r-1},E] = -1/2\Sigma_{j=p-q}^{N-1}[\omega_j[\omega_{N+p-r-1-j},E]]^{\tilde{m}}$$

$$+ \Sigma(\text{terms in } \omega_{j<N}) \qquad (3.25)$$

and so $\omega^{\tilde{m}}_{N+p-r-1}$ is determined. Hence ω_n is determined to all orders. So for any p, it is possible to choose the dynamical sector so that ω may be determined.

The gauge equation (3.1) determines $\omega_n^{\tilde{k}}$ non-locally, and so one would expect the equations of motion (2.20) to be non-local. This is true in general, but for systems in which the dynamical sector is local, the fundamental hierarchy are in fact local equations of motion (in any local gauge). To show this, let w be of the form

$$w = \exp\Sigma_{n=1}^{\infty}\lambda^{-n}w_n \tag{3.26}$$

and define the transformed gauge potentials

$$a_{n,g} = w^{-1}A_{n,g}w + w^{-1}w_{n,g} \tag{3.27}$$

Now choose

$$w_n = \omega_n \qquad 1\leq n\leq p-1 \tag{3.28a}$$

$$w_p^{\tilde{m}} = \omega_p^{\tilde{m}} \tag{3.28b}$$

i.e. ω,w coincide on the dynamical sector. Then (recalling (2.22b), and letting ϕ be the basic gauge), a_x takes the form

$$a_x = \lambda^p E + \Sigma_{n=1}^{\infty}\lambda^{-n}a_x^{-n} \tag{3.29}$$

while the transformed potentials of the fundamental hierarchy are of the form

$$a_{N,E} = \lambda^N E + \Sigma_{n=-\infty}^{N-p-1}\lambda^n a_{N,E}^n \tag{3.30}$$

(using (2.20b) with $\psi=1, g=E$). Now specify the remaining generators of w by choosing $a_x \in \underset{\sim}{k}$, i.e.

$$(w^{-1}Ew)^{\underset{\sim}{m}}_{n+p} = -(w^{-1}w_x)^{\underset{\sim}{m}}_{n} - \Sigma^{p-1}_{r=0}(w^{-1}A^r_x w)^{\underset{\sim}{m}}_{r+n} \quad \forall n \geq 1 \qquad (3.31)$$

and so

$$[w_{n+p}, E] = \Sigma(\text{terms in } w_{j<n+p}) \qquad (3.32)$$

Then $w^{\underset{\sim}{m}}_n$ $(n>p)$ is locally defined to all orders in terms of the dynamical sector. $w^{\underset{\sim}{k}}_n$ $(n \geq p)$ is left undetermined, and will be chosen to vanish.

In the gauge (3.27), the equations of motion of the fundamental hierarchy have the representation

$$\partial_x a_{N,E} - \partial_{N,E} a_x + [a_x, a_{N,E}] = 0 \qquad (3.33)$$

(from Eq.(2.12)). It will now be shown that $a_{N,E}$ may actually be written as

$$a_{N,E} = \lambda^N E + \Sigma^{\infty}_{n=1} \lambda^{-n} a^{-n}_{N,E} \qquad (3.34)$$

This is certainly the case if $N \leq p$ (from (3.30)). Suppose $N>p$. Substituting (3.29),(3.30), one may write the coefficient of λ^{N-n} in (3.33) as

$$\partial_x a^{N-n}_{N,E} + [E, a^{N-p-n}_{N,E}] + \Sigma^{n-p-1}_{r=1}[a^{-r}_x, a^{N-n+r}_{N,E}] = 0 \quad 1 \leq n \leq N \qquad (3.35)$$

i.e.

$$[E,\ a_{N,E}^{N-p-n}] = 0 \qquad 1\leq n\leq p \qquad (3.36a)$$

$$\partial_x a_{N,E}^{N-n} + [E,\ a_{N,E}^{N-p-n}] + \Sigma_{r=1}^{n-p-1}[a_x^{-r},\ a_{N,E}^{N-n+r}] = 0 \qquad p+1\leq n\leq N \qquad (3.36b)$$

Eq.(3.36a) shows that $a_{N,E}^{N-2p},\ldots,a_{N,E}^{N-p-1}$ $\underset{\sim}{k}$. Then the $\underset{\sim}{m}$ component of Eq.(3.36b) becomes

$$[E,a_{N,E}^{N-p-n}] = -\partial_x(a_{N,E}^{N-n})^{\underset{\sim}{m}} - \Sigma_{r=1}^{n-2p-1}[a_x^{-r},(a_{N,E}^{N-n+r})^{\underset{\sim}{m}}] \qquad p+1\leq n\leq N \qquad (3.37)$$

i.e.

$$[E,\ a_{N,E}^{N-p-n}] = 0 \qquad p+1\leq n\leq \min(N,2p) \qquad (3.38)$$

and so the upper bound on the summation in (3.37) is reduced to n-3p-1. Repeating until min(N,mp) = N for some m, one finds that

$$a_{N,E}^{n} \in \underset{\sim}{k} \qquad -p\leq n\leq N-p-1 \qquad (3.39)$$

Then the $\underset{\sim}{k}$ component of Eq.(3.36b) is

$$\partial_x a_{N,E}^{N-n} = -\Sigma_{r=1}^{n-p-1}[a_x^{-r},\ a_{N,E}^{N-n+r}] \qquad p+1\leq n\leq N \qquad (3.40)$$

Putting n=p+1, one has

$$\partial_x a_{N,E}^{N-p-1} = 0 \tag{3.41}$$

so one may choose $a_{N,E}^{N-p-1} = 0$. Then, by induction, Eq.(3.40) allows one to choose

$$a_{N,E}^{N-n} = 0 \qquad p+1 \leq n \leq N \tag{3.42}$$

and so the assertion (3.34) is proved. Now invert the gauge transformation (3.27) to obtain

$$A_{N,E} = wa_{N,E}w^{-1} - w_{N,E}w^{-1}$$

$$= \lambda^N wEw^{-1} + \Sigma_{n=1}^{\infty}\lambda^{-n} wa_{N,E}^{-n}w^{-1} - w_{N,E}w^{-1} \tag{3.43}$$

Since $A_{N,E}$ is a polynomial in only positive powers of λ, one may equate coefficients to get

$$A_{N,E} = \Sigma_{n=0}^{N}\lambda^n(wEw^{-1})_{N-n} \tag{3.44a}$$

$$= \Sigma_{n=0}^{N}\lambda^n(\omega E\omega^{-1})_{N-n} \qquad \forall N \geq 0 \tag{3.44b}$$

(from (2.20b) with $\phi=1$), and so

$$wEw^{-1} = \omega E\omega^{-1} \tag{3.45}$$

(where the constants of integration in $\omega E\omega^{-1}$ are set to zero). The equations of motion of the fundamental hierarchy may be represented as

$$\partial_x A_{N,E} - \partial_{N,E} A_x + [A_x, A_{N,E}] = 0 \tag{3.46}$$

(from Eq.(2.12)). If the dynamical sector is local, then w (and hence $A_{N,E}$ (3.44a)) will be local, and so the equations of motion (3.46) are local. The associated Hamiltonian realization will consist of non-local Hamiltonians with a sub-class of local quantities $H_{N,E}$.

4. REALIZATIONS WITH p = 1

In the basic gauge, with p=1, the gauge potential (2.22b) is

$$A_x = \lambda E + [\omega_1, E] \tag{4.1}$$

For G=SU(2), this defines the AKNS system [7]. Since the Cartan subalgebra is one-dimensional, the equations of motion all belong to the central hierarchy.

Now, from (2.21c) with N=n=1 and ψ=1,

$$\partial_{-1,E}\overset{m}{\underset{\sim}{\omega_1}} = (-A^1_{-1,E})^{\overset{m}{\sim}}$$

$$= (-\Psi E \Psi^{-1})^{\overset{m}{\sim}} \tag{4.2}$$

(using (2.21b). In the basic gauge, one also has ((2.24a) with N=1,g=E)

$$\Psi_x \Psi^{-1} = -[\omega_1, E] \tag{4.3}$$

so that the equation of motion may be rewritten (with $\partial_t \equiv \partial_{-1,E}$) as

$$\partial_t(\Psi_x\Psi^{-1}) = [\Psi E\Psi^{-1}, E] \tag{4.4}$$

For G=SU(2), choose

$$E = 1/\sqrt{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.5}$$

$$[\omega_1, E] = \begin{pmatrix}0 & u\\ -u & 0\end{pmatrix} \tag{4.6}$$

where u is a real field. Then (noting (4.3)), Ψ may be written as

$$\Psi = \begin{pmatrix}\exp(i\beta)\cos(\alpha/2) & \exp(-i\beta)\sin(\alpha/2)\\ -\exp(i\beta)\sin(\alpha/2) & \exp(-i\beta)\cos(\alpha/2)\end{pmatrix} \tag{4.7}$$

where $\alpha_x=2u$ and $\beta_x=0$. Then Eq (4.4) becomes

$$\alpha_{xt} = \sin\alpha \tag{4.8}$$

which is the sine-Gordon equation.

Now consider the equation of motion with $\partial_t \equiv \partial_{2,E}$. Using (2.20a) one has

$$\partial_t\omega_1 = (\omega_{2,E}\omega^{-1})_1 = (\omega E\omega^{-1})_3 \tag{4.9}$$

and, since $\partial_x \equiv \partial_{1,E}$, Eq.(2.20a) gives

$$(\omega E \omega^{-1})_3 = (\omega_x \omega^{-1})_2$$

$$= \partial_x \omega_2 + 1/2[\omega_1, \partial_x \omega_1] \qquad (4.10)$$

(from Eq.(3.3a)). Then

$$\partial_t \overset{m}{\tilde{\omega}_1} = \partial_x \overset{m}{\tilde{\omega}_2} + 1/2[\overset{k}{\tilde{\omega}_1}, \partial_x \overset{m}{\tilde{\omega}_1}] + 1/2[\overset{m}{\tilde{\omega}_1}, \partial_x \overset{k}{\tilde{\omega}_1}] + 1/2[\overset{m}{\tilde{\omega}_1}, \partial_x \overset{m}{\tilde{\omega}_1}]^{\overset{m}{\sim}}$$

$$(4.11)$$

This may be evaluated using (3.8). One obtains

$$\partial_t[\omega_1, E] = \partial_{xx} \overset{m}{\tilde{\omega}_1} - [\partial_x \overset{m}{\tilde{\omega}_1}[\omega_1, E]]^{\overset{m}{\sim}} - 1/2[E[\overset{m}{\tilde{\omega}_1}[\overset{m}{\tilde{\omega}_1}[\omega_1, E]]]] \qquad (4.12)$$

For G=SU(2), one may choose

$$E = -i/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4.13a)$$

$$[\omega_1, E] = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \qquad (4.13b)$$

If $\underset{\sim}{g}$ is restricted to the compact or non-compact real form ($r = \pm q^*$) then Eq.(4.12) becomes

$$iq_t = q_{xx} \mp 2q|q|^2 \qquad (4.14)$$

which is the non-linear Schrödinger equation. For a symmetric Lie algebra $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$ the middle term of Eq.(4.12) will vanish, and the equation of motion (when $\underset{\sim}{g}$ is restricted to the compact or non-compact real form is the matrix form of the generalized non-linear Schrödinger equation discussed in [4]. The Hamiltonian realization associated with this system was constructed in [2,3].

Now consider the gauge choice ϕ K (the Lie group of $\underset{\sim}{k}$). Then (2.22b) with p=1 is

$$A_x = \lambda E + [\phi\omega_1\phi^{-1}, E] - \phi_x\phi^{-1} \tag{4.15}$$

For G=SU(2), choose

$$E = -1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.16a}$$

$$[\phi\omega_1\phi^{-1}, E] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.16b}$$

$$\phi_x\phi^{-1} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \tag{4.16c}$$

For $\partial_t \equiv \partial_{2,E}$ write

$$\phi_t\phi^{-1} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \tag{4.17}$$

The consistency condition (2.23) on ϕ becomes

$$u_t = v_x \tag{4.18}$$

and one also has (using (2.15b))

$$0 = \partial_t[\phi\omega_1\phi^{-1}, E]$$

$$= [[\phi_t\phi^{-1},\ \phi\omega_1\phi^{-1}]E] + [\phi\partial_t\omega_1\phi^{-1},\ E] \tag{4.19}$$

$$= [\phi_t\phi^{-1}[\phi\omega_1\phi^{-1},E]] - [\partial_x(\phi_x\phi^{-1}),\ \phi\overset{m}{\tilde{\omega}}_1\phi^{-1}]$$

$$+ [\phi_x\phi^{-1}[\phi_x\phi^{-1},\ \phi\overset{m}{\tilde{\omega}}_1\phi^{-1}]] \tag{4.20}$$

(using (A.7),(4.12),(4.16b), and using (2.15b) to rewrite $\phi\partial_{xx}\overset{m}{\tilde{\omega}}_1\phi^{-1}$). Substituting (4.16),(4.17), this becomes

$$v = u_x + u^2 \tag{4.21}$$

Then the equation of motion is given by (4.18)

$$u_t = u_{xx} + 2uu_x \tag{4.22}$$

which is the Burgers equation. ϕ plays the rôle of the Cole-Hopf transformation [8,9].

In the principal gauge, with p=1, the gauge potential is

$$A_x = \lambda\phi E\phi^{-1} \tag{4.23}$$

(from (2.22b),(2.27a)). The equation of motion with $\partial_t \equiv \partial_{2,E}$ is

$$\partial_t(\phi E\phi^{-1}) = [\phi_{2,E}\phi^{-1},\ \phi E\phi^{-1}] \qquad \text{(by (2.15b))}$$

$$= \phi[(\omega E\omega^{-1})_2, E]\phi^{-1} \qquad \text{(by (2.27a))}$$

$$= \phi[\partial_x\omega_1, E]\phi^{-1} \qquad \text{(by (2.22a))}$$

$$= \phi\partial_x(\phi^{-1}\phi_x)\phi^{-1} \qquad \text{(by (2.27a))}$$

$$= \partial_x(\phi_x\phi^{-1}) \qquad \text{(by (2.15a))} \qquad (4.24)$$

Now let $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$ be a symmetric algebra, with $(\mathrm{ad}E)^2 = -I$ on $\underset{\sim}{m}$. Then (since $\phi^{-1}\phi_x \in \underset{\sim}{m}$, by (2.27a)), Eq.(4.24) may be rewritten

$$\partial_t(\phi E\phi^{-1}) = -\partial_x(\phi[[\phi^{-1}\phi_x, E], E]\phi^{-1})$$

$$= \partial_x[\phi E\phi^{-1}, \partial_x(\phi E\phi^{-1})] \qquad (4.25)$$

Writing $\phi E\phi^{-1} = S$, this becomes

$$\partial_t S = [S, S_{xx}] \qquad (4.26)$$

which is the generalized Heisenberg ferromagnet equation [4]. The gauge equivalence of the non-linear Schrödinger and Heisenberg ferromagnet equations was first demonstrated in [10].

5. REALIZATIONS WITH p = 2

It was shown in Section 3 (Eq (3.11)) that for p=2 the dynamical sector must satisfy

$$\partial_x \omega_1^{\mathfrak{k}} = 1/2[\omega_1^{\mathfrak{m}}[\omega_2,E]]^{\mathfrak{k}} + 1/2[\omega_2^{\mathfrak{m}}[\omega_1,E]]^{\mathfrak{k}} + 1/6[\omega_1[\omega_1[\omega_1,E]]]^{\mathfrak{k}} \tag{5.1}$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a symmetric algebra with $(\mathrm{ad}E)^2 = -I$ on $\mathfrak{m}$. Then one may choose

$$\omega_1^{\mathfrak{k}} = 0 \tag{5.2a}$$

$$\omega_2^{\mathfrak{m}} = 0 \tag{5.2b}$$

Now one may consistently choose

$$\omega_n \in \mathfrak{m} \qquad (n \text{ odd}) \tag{5.3a}$$

$$\omega_n \in \mathfrak{k} \qquad (n \text{ even}) \tag{5.3b}$$

To see why this is so, note that the commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \tag{5.4a}$$

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \tag{5.4b}$$

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \tag{5.4c}$$

may be realized by $\mathfrak{k} \to \{2n\}$, $\mathfrak{m} \to \{2n+1\}$ $(n \in \mathbb{Z})$ and $[a,b] \to a+b$. Then by induction on n in (3.3) one obtains

$$(\omega_\mu \omega^{-1})_n \in \begin{cases} \underset{\sim}{m} & (n \text{ odd}) \\ \underset{\sim}{k} & (n \text{ even}) \end{cases} \tag{5.5a}$$

$$(\omega k \omega^{-1})_n \in \begin{cases} \underset{\sim}{m} & (n \text{ odd}) \\ \underset{\sim}{k} & (n \text{ even}) \end{cases} \qquad \forall\, k \in \underset{\sim}{k} \tag{5.5b}$$

$$(\omega m \omega^{-1})_n \in \begin{cases} \underset{\sim}{k} & (n \text{ odd}) \\ \underset{\sim}{m} & (n \text{ even}) \end{cases} \qquad \forall\, m \in \underset{\sim}{m} \tag{5.5c}$$

and so the gauge equation

$$(\omega_x \omega^{-1})_n = (\omega E \omega^{-1})_{n+2} \tag{5.6}$$

is consistent. Then (5.5) and (2.20a) imply that

$$\partial_{N,k}\omega = 0 \qquad (N \geq 0 \text{ odd},\ k \in \underset{\sim}{k}) \tag{5.7a}$$

$$\partial_{N,m}\omega = 0 \qquad (N \geq 0 \text{ even},\ m \in \underset{\sim}{m}) \tag{5.7b}$$

Now let $\phi \in K$. Then the gauge potential (2.22b), with the choice (5.2), becomes

$$A_x = \lambda^2 E + \lambda[\phi\omega_1\phi^{-1},E] + 1/2[\phi\omega_1\phi^{-1}[\phi\omega_1\phi^{-1},E]] - \phi_x\phi^{-1} \tag{5.8}$$

Now consider the equation of motion with $\partial_t \equiv \partial_{4,E}$. From (2.20a)

$$\begin{aligned} \partial_t\omega_1 &= (\omega E\omega^{-1})_5 = (\omega_x\omega^{-1})_3 \qquad \text{(using (2.22a))} \\ &= \partial_x\omega_3 + 1/2[\omega_2, \partial_x\omega_1] + 1/2[\omega_1, \partial_x\omega_2] \\ &\quad + 1/6[\omega_1[\omega_1, \partial_x\omega_1]] \qquad \text{(from (3.3a))} \end{aligned} \tag{5.9}$$

To obtain an explicit expression, ω_2 and ω_3 must be determined from the gauge equation (5.6). Putting n=1, one obtains

$$[\omega_3,E] = \partial_x\omega_1 -1/2[\omega_2[\omega_1,E]] - 1/6[\omega_1[\omega_1[\omega_1,E]]] \tag{5.10}$$

while n=2 gives (using (A.8))

$$\partial_x\omega_2 = [\omega_1[\omega_3,E]] + 1/6[\omega_2[\omega_1[\omega_1,E]]] +1/6[\omega_1[\omega_2[\omega_1,E]]]$$

$$+1/24[\omega_1[\omega_1[\omega_1[\omega_1,E]]]] - 1/2[\omega_1, \partial_x\omega_1] \tag{5.11}$$

Substituting (5.10) and using (A.10) this becomes

$$\partial_x\omega_2 = 1/2[\omega_1, \partial_x\omega_1] - 1/8[\omega_1[\omega_1[\omega_1[\omega_1,E]]]] \tag{5.12}$$

Then the equation of motion (5.9) is

$$\partial_t\omega_1 = [E, \partial_{xx}\omega_1] + 2/3[\omega_1[\omega_1, \partial_x\omega_1]] - 1/6\partial_x[E[\omega_1[\omega_1[\omega_1,E]$$

$$- 1/8[\omega_1[\omega_1[\omega_1[\omega_1[\omega_1,E]]]]] \tag{5.13}$$

Now, (using (2.15b), (A.7)),

$$\partial_t A_x^1 = \partial_t[\phi\omega_1\phi^{-1}, E]$$

$$= [\phi_t\phi^{-1}[\phi\omega_1\phi^{-1}, E]] + [\phi(\partial_t\omega_1)\phi^{-1}, E] \tag{5.14}$$

Substituting (5.13), and using (2.15b) to rewrite $\phi(\partial_x^n\omega_1)\phi^{-1}$, this becomes

$$\partial_t A_x^1 = [\phi_t\phi^{-1}, A_x^1] + [E, \partial_{xx}A_x^1] + [\partial_x(\phi_x\phi^{-1})[A_x^1,E]]$$

$$+ 2[\phi_x\phi^{-1}[\partial_x A_x^1,E]] - [\phi_x\phi^{-1}[\phi_x\phi^{-1}[A_x^1,E]]]$$

$$+ 2/3[A_x^1[A_x^1, \partial_x A_x^1]] - 1/6\partial_x[E[A_x^1[A_x^1[A_x^1,E]]]]$$

$$+ 1/6[\phi_x\phi^{-1}[E[A_x^1[A_x^1[A_x^1,E]]]]] + 2/3[A_x^1[A_x^1[A_x^1, \phi_x\phi^{-1}]]]$$

$$- 1/8[A_x^1[A_x^1[A_x^1[A_x^1[A_x^1,E]]]]] \tag{5.15}$$

$$\equiv [\phi_t\phi^{-1}, A_x^1] + F \tag{5.16}$$

Now choose

$$\phi_x\phi^{-1} = \alpha[A_x^1[A_x^1,E]] \tag{5.17}$$

Then $\phi_t\phi^{-1}$ is defined by the consistency condition

$$\partial_x(\phi_t\phi^{-1}) = \partial_t(\phi_x\phi^{-1}) + [\phi_x\phi^{-1}, \phi_t\phi^{-1}] \tag{5.18}$$

which may be rewritten using (5.16) (and (A.7,8,10)) as

$$\partial_x(\phi_t\phi^{-1}) = 2\alpha[\partial_t A_x^1[A_x^1,E]] - \alpha[\phi_t\phi^{-1}[A_x^1[A_x^1,E]]$$

$$= 2\alpha[F[A_x^1,E]] \tag{5.19}$$

This becomes (using (A.7,8,12,13,16))

$$\partial_x(\phi_t\phi^{-1}) = 2\alpha[A_x^1, \partial_{xx}A_x^1] + (4\alpha^2+\alpha)[A_x^1[\partial_x A_x^1[A_x^1[A_x^1,E]]]]$$

$$+ (4\alpha^2-2\alpha)[A_x^1[A_x^1[\partial_x A_x^1[A_x^1,E]]]] \qquad (5.20)$$

i.e. (using (A.15))

$$\phi_t\phi^{-1} = 2\alpha[A_x^1, \partial_x A_x^1] + (\alpha^2 + \alpha/4)[A_x^1[A_x^1[A_x^1[A_x^1,E]]]]$$

$$+ (4\alpha^2 - 2\alpha)\partial^{-1}[A_x^1[A_x^1[\partial_x A_x^1[A_x^1,E]]]] \qquad (5.21)$$

Substituting this and (5.17) into (5.15) gives the equation of motion:

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] + (2\alpha - 4\alpha^2)[A_x^1,\partial^{-1}[A_x^1[A_x^1[\partial_x A_x^1[A_x^1,E]]]]$$

$$+ 1/2[E[\partial_x A_x^1[A_x^1[A_x^1,E]]]] + (4\alpha - 1)[E[A_x^1[\partial_x A_x^1[A_x^1,E]]]]$$

$$+ (\alpha/4 - 1/8)[A_x^1[A_x^1[A_x^1[A_x^1[A_x^1,E]]]]] \qquad (5.22)$$

(where (A.12,13,16) have been used). This equation becomes local for two values of α. Putting $\alpha=0$ (the basic gauge) gives a quintic non-linear Schrödinger equation. Putting $\alpha = 1/2$ (the principal gauge), one obtains

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] + 1/2\partial_x[E[A_x^1[A_x^1[A_x^1,E]]]] \qquad (5.23)$$

When g is restricted to the compact or non-compact real form, the components of (5.23) give the derivative non-linear Schrödinger equation of Fordy [11].

For $\underset{\sim}{g} = sl(2,\mathbb{C})$, one can write

$$A_x^1 = qe_+ + re_- \tag{5.24}$$

Then one can check that for this case

$$[A_x^1[A_x^1[\partial_x A_x^1[A_x^1,E]]]] = 4\partial_x[A_x^1[A_x^1[A_x^1[A_x^1,E]]]] \tag{5.25}$$

and so $\phi_t\phi^{-1}$ (5.20) is local for all α. The equation of motion (5.22) is then

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] - (\alpha - 1/2)(\alpha - 1/4)[A_x^1[A_x^1[A_x^1[A_x^1[A_x^1,E]]]]]$$

$$+ 1/2[E[\partial_x A_x^1[A_x^1[A_x^1,E]]]] + (4\alpha - 1)[E[A_x^1[\partial_x A_x^1[A_x^1,E]]]] \tag{5.26}$$

Putting $\alpha = 1/4$, this becomes

$$\partial_t A_x^1 = [E, \partial_{xx}A_x^1] + 1/2[E[\partial_x A_x^1[A_x^1[A_x^1,E]]]] \tag{5.27}$$

The restriction to the compact or non-compact real form ($r=\pm q^*$) gives the derivative non-linear Schrödinger equation of Chen et al [12].

The gauge potentials of the zero curvature representation are easily found. Substituting (5.17) into (5.8) gives

$$A_x = \lambda^2 E + \lambda A_x^1 + (1/2 - \alpha)[A_x^1[A_x^1, E]] \tag{5.28}$$

while from (2.20b)

$$A_t = \lambda^4 E + \lambda^3[\phi\omega_1\phi^{-1}, E] + \lambda^2[\phi\omega_1\phi^{-1}[\phi\omega_1\phi^{-1}, E]]$$

$$+ \lambda\phi(\partial_x\omega_1)\phi^{-1} + \phi(\partial_x\omega_2 + 1/2[\omega_1, \partial_x\omega_1])\phi^{-1} - \phi_t\phi^{-1} \qquad (5.29)$$

(where (5.6) has been used). This becomes (using (5.11),(5.17),(5.21))

$$A_t = \lambda^4 E + \lambda^3 A_x^1 + \lambda^2[A_x^1[A_x^1, E]]$$

$$+ \lambda([E, \partial_x A_x^1] + \alpha[E[A_x^1[A_x^1[A_x^1, E]]]])$$

$$+ (1-2\alpha)[A_x^1, \partial_x A_x^1] - (\alpha^2 - 3\alpha/4 + 1/8)[A_x^1[A_x^1[A_x^1[A_x^1, E]]]]$$

$$- (4\alpha^2 - 2\alpha)\partial^{-1}[A_x^1[A_x^1[\partial_x A_x^1[A_x^1, E]]]] \qquad (5.30)$$

6. DISCUSSION

Various authors have used gauge invariance to investigate the conserved quantities of integrable dynamical systems, such as the nonlinear sigma model in [13] (where the gauge potentials are rational functions of λ), and the Toda equation in [14]. The gauge transformation w (3.31) is a generalization of the local transformation used in [14]. Non-local conserved quantities were constructed for the non-linear sigma model in [15], and these are associated with infinitesimal transformations which realize the "positive half" of a centre-free Kac-Moody algebra [16]. However, the charges themselves do not form an algebra [17]. The infinitesimal symmetries of the SU(2) non-linear Schrödinger equation were investigated in [18], using the gauge transformation to the Heisenberg ferromagnet. Only the "positive" subalgebra was realized non-trivially, since the gauge transformation is in fact the principal gauge (4.23), which is trivial in the negative part of the algebra (2.27b).

Much work has been done by the Kyoto group [19] on the construction of soliton solutions for non-linear dynamical systems using vertex operators. The construction of solitons within the formalism of this paper, and the connection with the work of the Japanese authors, will be dealt with in a subsequent paper.

Another topic which will be investigated further is the construction of the Hamiltonian realizations. One must first define the Poisson bracket between matrix elements of the dynamical sector (with different values of the spectral parameter), i.e.

$$\{A_x(x_1,\lambda_1) \underset{,}{\otimes} A_x(x_2,\lambda_2)\} \tag{6.1}$$

(where the notation of [20] is used). Such objects have been studied in connection with the "r-matrix" [21] and Yang-Baxter equation [22].

APPENDIX

Let $\underset{\sim}{g}$ be a semisimple Lie algebra over a field P of characteristic zero. (By "semisimple" it is meant that $\underset{\sim}{g}$ has a symmetric bilinear form (,) which is non-degenerate). A Cartan subalgebra of $\underset{\sim}{g}$ is a maximal abelian subalgebra $\underset{\sim}{h}$ with the property that, for all $h \in \underset{\sim}{h}$, adh is a semisimple endomorphism of $\underset{\sim}{g}$. (Recall that $(\mathrm{ad}h)g \equiv [h,g]$, and a semisimple endomorphism is one for which the roots of the minimal polynomial are all distinct). Every semisimple Lie algebra has at least one Cartan subalgebra. Choose one, and let E be an element of it. Define

$$\underset{\sim}{k} = \{g \in g:\ [E,g] = 0\} \tag{A.1a}$$

$$\underset{\sim}{l} = \{g \in \underset{\sim}{g}:\ (\mathrm{ad}E)^n g = 0\} \tag{A.1b}$$

$$\underset{\sim}{m} = [E,\underset{\sim}{g}] \tag{A.1c}$$

Being semisimple, adE is diagonalizable over P (or its algebraic closure), and hence $\underset{\sim}{k} = \underset{\sim}{l}$. Then $\underset{\sim}{k} \cap \underset{\sim}{m} = 0$. Also, the fact that $\underset{\sim}{g}$ is semisimple, together with the relation $0 = (\underset{\sim}{g},[\underset{\sim}{k},E]) = ([E,\underset{\sim}{g}],\underset{\sim}{k})$, implies that $\dim \underset{\sim}{k} + \dim \underset{\sim}{m} = \dim \underset{\sim}{g}$; i.e. $\underset{\sim}{g} = \underset{\sim}{k} \oplus \underset{\sim}{m}$ (direct sum as vector spaces). If there exist $k \in \underset{\sim}{k}$, $m \in \underset{\sim}{m}$ such that $[k,m] \in \underset{\sim}{k}$ $(\neq 0)$, then (writing $m = [E,m']$)

$$0 = [E[k[E,m']]] = [E[E[k,m']]] \tag{A.2}$$

(by the Jacobi identity) and so

$$0 = (\underset{\sim}{g},[E[E[k,m']]]) = ([\underset{\sim}{g},E],[E[k,m']]) \tag{A.3}$$

However,

$$(\underset{\sim}{k},[E[k,m']]) = ([\underset{\sim}{k},E],[k,m']) = 0 \tag{A.4}$$

i.e. $[E[k,m']] = [k,m]$ is orthogonal to $\underset{\sim}{g}$, which contradicts the fact that $\underset{\sim}{g}$ is semisimple. So one has the relations

$$[\underset{\sim}{k},\underset{\sim}{m}] \subset \underset{\sim}{m} \qquad [\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \tag{A.5}$$

(the latter follows from the Jacobi identity on $[E[\underset{\sim}{k},\underset{\sim}{k}]]$). The corresponding Lie group G/K is called a reductive homogeneous space. For certain algebras it is possible to choose E so that one has the additional relation

$$[\underset{\sim}{m},\underset{\sim}{m}] \subset \underset{\sim}{k} \tag{A.6}$$

then G/K is called a symmetric space, and g is called a symmetric Lie algebra. An important example arises when adE is a complex structure on $\underset{\sim}{m}$ (i.e. $(\mathrm{adE})^2 = -I$ on $\underset{\sim}{m}$). For further details, see [23].

In the general (reductive) case, it is clear (using the Jacobi identity) that

$$[k[m,E]] = [[k, m],E] \qquad \forall k \in \underset{\sim}{k},\ m \in \underset{\sim}{m} \tag{A.7}$$

$$[m_1[m_2,E]]^{\underset{\sim}{k}} = [m_2[m_1,E]^{\underset{\sim}{k}} \qquad \forall\ m_1,m_2 \in \underset{\sim}{m} \tag{A.8}$$

(where the superscript indicates the k component). Also, for all $k \in \underset{\sim}{k}$, $m \in \underset{\sim}{m}$,

$$[k[m[m,E]]] = [m[k[m,E]]] - [[m,E],[k,m]]$$

$$= 2[m[k[m,E]]] + [E[m[k,m]]] \qquad \text{(A.9)}$$

i.e.

$$[k[m[m,E]]]^{\underset{\sim}{k}} = 2[m[k[m,E]]]^{\underset{\sim}{k}} = 2[[E,m],[k,m]]^{\underset{\sim}{k}} \qquad \text{(A.10)}$$

From now on it is assumed that E can be (and is) chosen so that $(\mathrm{ad}E)^2 = \kappa^2$ (constant) on $\underset{\sim}{m}$ (i.e. the symmetric case). Notice that

$$[[E,m_1],[E,m_2]] = -\kappa^2[m_1,m_2] \qquad \forall m_1,m_2 \in \underset{\sim}{m} \qquad \text{(A.11)}$$

Also

$$[m[E[m[m[m,E]]]]] = [m[[m[m,E]],[m,E]]] \qquad \text{(by (A.7))}$$

$$= 1/2[[m[m,E]],[m[m,E]]] \qquad \text{(by (A.10))}$$

$$= 0 \qquad \text{(A.12)}$$

and

$$[m[E[m[m[m[m,E]]]]]]$$

$$= [m[[E,m],[m[m[m,E]]]]] \qquad \text{(by (A.7))}$$

$$= [m[[[E,m]m],[m[m,E]]]] + [m[m[[E,m],[m[m,E]]]]]$$

(by the Jacobi identity)

$$= [[m[m,E]],[m[m[m[m,E]]]]]$$

(by the Jacobi identity and (A.8),(A.12))

$$= [m[[m,E],[m[m[m[m,E]]]]]] - [[m,E],[m[m[m[m[m,E]]]]]]]$$

(by the Jacobi identity)

$$= -2[m[E[m[m[m[m[m,E]]]]]]] \qquad \text{(by (A.7),(A.8))}$$

$$= 0 \qquad \text{(A.13)}$$

Now let $m,m' \in \underset{\sim}{m}$. Then the Jacobi identity implies

$$[m'[m[m[m,E]]]]$$

$$= [[m',m],[m[m,E]]] + [m[m'[m[m,E]]]]$$

$$= 2[m[[m',m],[m,E]]]] + [m[m'[m[m,E]]]] \qquad \text{(by (A.10))}$$

$$= 3[m[m'[m[m,E]]]] - 2[m[m[m'[m,E]]]] \qquad \text{(A.14)}$$

In particular, if ∂ is a differential operator then (using (A.8))

$$\partial[m[m[m[m,E]]]] = 4[m[\partial m[m[m,E]]]] \qquad \text{(A.15)}$$

Lastly, since $(\mathrm{ad}E)^2 = \kappa^2$ on $\underset{\sim}{m}$, one has for all $m,m' \in \underset{\sim}{m}$:

$$[m[m,m']] = \kappa^{-2}[E[E[m[m,m']]]]$$

$$= \kappa^{-2}[E[[E,m],[m,m']]] \qquad \text{(by (A.7))}$$

$$= \kappa^{-2}[E[m[m'[m,E]]]] - \kappa^{-2}[E[m'[m[m,E]]]] \qquad \text{(A.16)}$$

REFERENCES

[1] Zakharov V.E., Shabat, A.B.: Sov. Phys. JETP 34,62 (1972)

[2] Crumey, A.D.W.B.: Local and non-local conserved quantities for generalized non-linear Schrödinger equations. Commun. Math. Phys. 108, 631-646 (1987) (Chapter 2 of this thesis)

[3] Crumey, A.D.W.B.: Kac-Moody symmetry of generalized non-linear Schrödinger equations. Commun. Math. Phys. 111, 167-179 (1987) (Chapter 3 of this thesis)

[4] Fordy, A.P., Kulish, P.P.: Nonlinear Schrödinger equations and simple Lie algebras. Commun. Math. Phys. 89, 427-443 (1983)

[5] Gervais, J.-L.: Gauge invariance over a group as the first principle of interacting string dynamics. LPTENS preprint 86/1 (1986)

[6] Goddard, P., Olive, D.: Kac-Moody and Virasoro algebras in relation to quantum physics. DAMTP 86-5 (1986)

[7] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Non-linear equations of physical significance. Phys. Rev. Lett. 31, 125-127 (1973)

[8] Hopf, E.: The partial differential equation $u_t + uu_x = \delta u_{xx}$ Comm. Pure and Appl. Math. 3, 201-230 (1950)

[9] Cole, J.D.: On a quasi-linear parabolic equation occuring in aerodynamics. Quart. Appl. Math. 9, 225-236 (1951)

[10] Zakharov, V.E., Takhtadzhyan, L.A.: Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet. Theor. Math. Phys. 38, 17 (1979)

[11] Fordy, A.: Derivative non-linear Schrödinger equations and Hermitian symmetric spaces. J. Phys. A: Math Gen 17, 1235-1245 (1984)

[12] Chen, H.H.. Lee, Y.C., Lin, C.S.: Phys. Scr. 20 490-492

[13] Eichenherr, H., Forger, M.: Higher local conservation laws for nonlinear sigma models on symmetric spaces. Commun. Math. Phys. 82, 227-255 (1981)

[14] Olive, D., Turok, N.: Local conserved densities and zero curvature conditions for Toda lattice field theories. Nucl. Phys. B257[FS14], 277 (1985)

[15] Luscher, M., Pohlmeyer, K.: Scattering of massless lumps and non-local charges in the two-dimensional classical non-linear σ-model. Nucl. Phys. B137, 46-54 (1978)

[16] Dolan, L: Kac-Moody algebra is hidden symmetry of chiral models. Phys. Rev. Lett. 47, 1371-1374 (1981)

[17] de Vega, H.J., Eichenherr, H., Maillet, J.M.: Classical and quantum algebras of non-local charges in σ-models. Commun. Math. Phys. 92 507-524 (1984)

[18] Eichenherr, H.: Symmetry algebras of the Heisenberg model and the non-linear Schrödinger equation. Phys. Lett. 115B, 385-38 (1982)

[19] For a review, see: Jimbo, M., Miwa, T.: Solitons and infinite dimensional Lie algebras. Publ. RIMS, Kyoto Univ. 19, 943-1001 (1983)

[20] Faddeev, L.D. In: Recent advances in field theory and statistical mechanics, Les Houches, Session 39, Elsevier Sci. Publ.563-608 (1984)

[21] Sklyanin, E.K.: Doklady AN SSSR 244, 1337 (1979)

[22] Kulish, P.P., Sklyanin, E.K.: On solutions of the Yang Baxter equation. Zap. Nauch. Semin. LOMI 95, 129 (1980)

CHAPTER 5: CONCLUDING REMARKS

The object of the work presented here has been to construct the algebra of conserved quantities for classical integrable systems which have zero curvature representations. It has been seen that this problem amounts to the construction of "dynamical" (Poisson bracket) realizations of centre free Kac-Moody algebras.

Although the formal linearization of these systems was given, the construction of explicit solutions was not carried out. This problem should make it possible to establish the connection between the present formalism and the work initiated by the Kyoto group [1] on the KP-hierarchy.

To see how these ideas may be linked, recall that the starting point in Chapter 4 was a family of operators $\partial_{n,g}$ and a "zero-curvature" equation which is the consistency condition for the system of equations:

$$(\partial_{n,g} + A_{n,g})\Phi = 0 \qquad (1)$$

where one could choose

$$A_{n,g} = \lambda^n g \qquad (2)$$

The dynamical equations of interest arose from the central hierarchy, corresponding to g = E. Then (1) has the solution

$$\Phi = \exp(-\Sigma_{n=-\infty}^{\infty}\lambda^n t_{n,E}E) \qquad (3)$$

The gauge potentials of the central hierarchy were obtained by applying the transformation $\Psi\Omega$ or $\psi\omega$, i.e. the system (1) becomes

$$0 = (\partial_{n,E} + \lambda^n \psi\omega E\omega^{-1}\psi^{-1} - (\psi\omega)_n \omega^{-1}\psi^{-1})\psi\omega \exp(-\Sigma_{n=-\infty}^{\infty}\lambda^n t_{n,E}E) \quad (4)$$

(and similarly with $\psi\omega$ replaced by $\Psi\Omega$). If one restricts $t_{n,E}$ to the fundamental hierarchy ($n \geq 0$) then the group valued solution becomes

$$w = \psi\omega \exp(-\Sigma_{n=0}^{\infty}\lambda^n t_{n,E}E) \quad (5)$$

This is (essentially) the "wave function" of the Kyoto group. In their formalism, one solves the dynamical system by relating w to the "τ function". Here, one notes that there are actually two expressions for the "wave function" (the other has $\psi\omega$ replaced by $\Psi\Omega$), which differ by a factor which is a polynomial in λ with constant coefficients. The dynamical solution might be obtainable if it is possible to "match" these two expressions (infinite polynomials). This could connect with the Riemann-Hilbert approach to integrable systems [2].

The τ function is usually regarded as a determinant on an infinite dimensional Grassmanian, and this interpretation has recently aroused interest within the context of string theory [3]. It may seem a pure coincidence that the mathematics of integrable systems (Kac-Moody algebras and their groups) should play a role there; on the other hand, integrable systems provide an extremal class (totally ordered, as opposed to totally disordered sytems), and one might expect Nature (or physicists) to favour one or other extremum at the fundamental level. However, the Grassmanian approach does not seem to offer any new insight into the physical principle (e.g. symmetry) which might underlie string theory. It is interesting that the present formalism is expressed in terms of current algebras (which provided the original approach to string theory), and uses the concept of "internal symmetry", which is of such importance in contemporary physics. If any deeper physical significance is to be attached to the dynamical realizations of Kac-Moody algebras, then two important problems should be considered. First, one should clarify the precise definition of "totally ordered" systems. It has been seen that the definitions of complete integrability become ambiguous in the infinite dimensional (field theory) case; given infinitely many conserved quantities it is not always clear whether one really has "enough". The systems which have been considered here seem to be worthy of the description "totally ordered", but there may be other equally worthy sytems which do not arise as realizations of Kac-Moody algebras. A precise definition of the concept of integrability applicable to field theories may prove to be equivalent to some kind of symmetry principle, which may be of relevance in fundamental physics. (The "gauge principle" of physics was already generalized in Chapter 4, where the directions of the original "space" were non-commuting). The second problem is to extend the formalism to quantum systems. This could probably be carried out in the manner of BRS quantization [4], which again is of importance in current approaches to string theory.

One could synthesize the two questions into the single problem of finding an appropriate way to discuss "integrability" for quantum field theories. These matters are the subject of current investigation.

REFERENCES

[1] For reviews, see: Jimbo, M., Miwa, T.: Solitons and infinite dimensional Lie algebras. Publ. RIMS, Kyoto Univ. 19, 943-1001 (1983); and Segal, G., Wilson, G.: Loop groups and equations of KdV type. Publ. I.H.E.S. 61, 5-65 (1985)

[2] Zakharov, V.E., Shabat, A.B.: Funct. Anal. and Appl. 13, 13 (1979)

[3] Alvarez-Gaumé, L., Gomez, C.: New methods in string theory. CERN preprint TH.4775/87 (1987)

[4] Bogojevic, A.R., Hlousek, Z.: BRST quantization of Kac-Moody algebras. Brown University preprint HET-596 (1986)